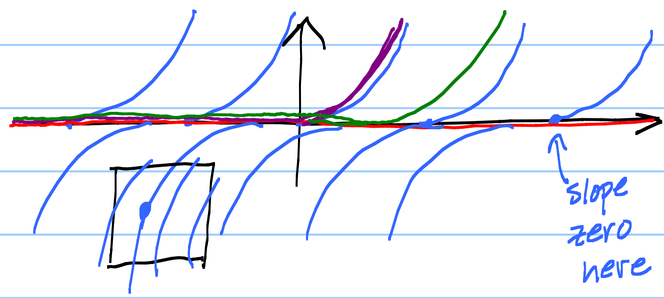
Lecture 6 Applications

HW05 due tonight

EX: $\frac{dy}{dx} = \sqrt{|y|}$ ← f continuous, but $\frac{\partial f}{\partial y}$ not along x-axis



Sol^{n}s exist, but are not unique.

∞ many sol^{n}s with $y(0)=0$

Newton's law of cooling

Prob: Hot rock in ocean

$T(t)$ = temp rock at time

T_0 = temp of ocean

$\frac{dT}{dt}$ is prop to $(T_0 - T)$

$$\boxed{\frac{dT}{dt} = k\,(T_0 - T)}$$

Separable ... and linear.

Advice: If linear, use linear method.

$$\frac{dT}{dt} + \underbrace{k}_{p(t)}\,T = k\,T_0$$

1 in front. Standard form ✓

$$u(t) = e^{\int p(t)\,dt} = e^{\int k\,dt} = e^{kt}$$

$$\underbrace{e^{kt}\left[T' + kT\right]}_{\left[e^{kt} T\right]'} = kT_0 e^{kt}$$

$$e^{kt} T = \int kT_0 e^{kt}\, dt = kT_0 \tfrac{1}{k} e^{kt} + C$$

$$\boxed{T(t) = T_0 + Ce^{-kt}}$$

Initial condition: know $T(0)$

$$T(0) = T_0 + Ce^{-k\cdot 0}$$

$$\boxed{C = T(0) - T_0}$$

$$\boxed{T(t) = T_0 + (T(0) - T_0)e^{-kt}}$$

See $\lim_{t\to\infty} T(t) = T_0$

Prob: Temp $40°$. Find stiff in yard. $65°$

When was time of death?

$$T(t) = 40^0 + (\underbrace{98.6}_{T(0)} - \underbrace{40}_{T_0})e^{-kt}$$

$t=0$ moment of death. Find it at t_F.

Know $T(0) = 98.6$

$$T(t_F) = 40 + 58.6\, e^{-kt_F} \overset{\text{know}}{=} 65$$

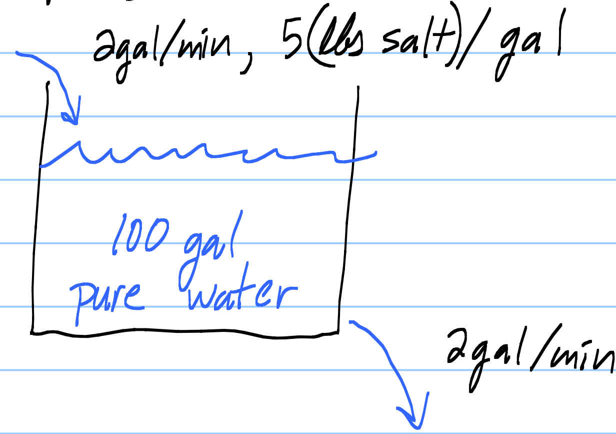
Hmmm. Can solve for kt_F here.

Aha! Wait an hour and measure temp again!

$$\left.\begin{array}{ll} \text{know} & kt_F \\ \text{get} & k(t_F+1) \end{array}\right\} \text{ two eqns, two unknowns}$$

<u>Prob</u>: Mixing problems

"well mixed"



$S(t)$ = lbs of salt in tank at time t

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$(2\text{gal/min})(5\text{lbs/gal}) - (2\text{gal/min}) \cdot \underbrace{\left[\frac{S(t)}{100 \text{ gals}}\right]}_{\text{depends on } S(t)!}$$

$$\frac{\cancel{\text{gal}}}{\text{min}} \cdot \frac{\text{lbs}}{\cancel{\text{gal}}}$$

Linear:
$$\frac{dS}{dt} + \frac{1}{50} S = 10$$

$$u = e^{\int \frac{1}{50} dt} = e^{t/50}$$

$$\underbrace{e^{t/50}\left(S' + \frac{1}{50} S\right)}_{\left[e^{t/50} S\right]'} = 10\, e^{t/50}$$

$$e^{t/50} S = \int 10\, e^{t/50}\, dt = 10 \cdot \frac{1}{\left(\frac{1}{50}\right)} e^{t/50} + C$$

$$S(t) = 500 + Ce^{-t/50} \leftarrow \frac{C}{e^{t/50}} = Ce^{-t/50}$$

Know $S(0) = 0$ ["pure" at time 0]

$$S(0) = 500 + Ce^{0} \overset{\text{want}}{=} 0 \qquad C = 500$$

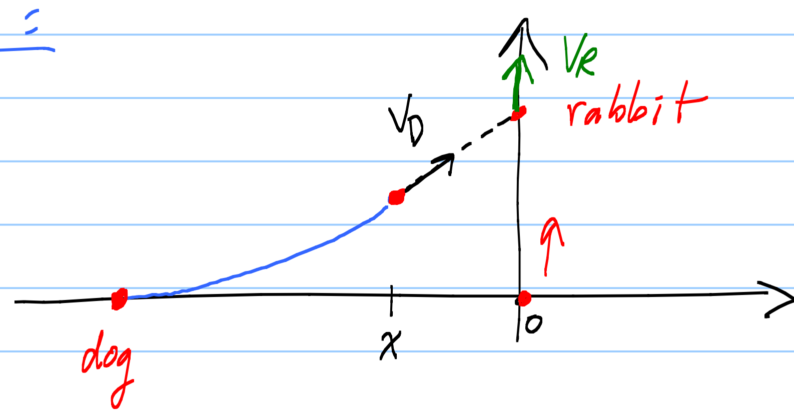
$$\boxed{S(t) = 500\left(1 - e^{-t/50}\right)}$$

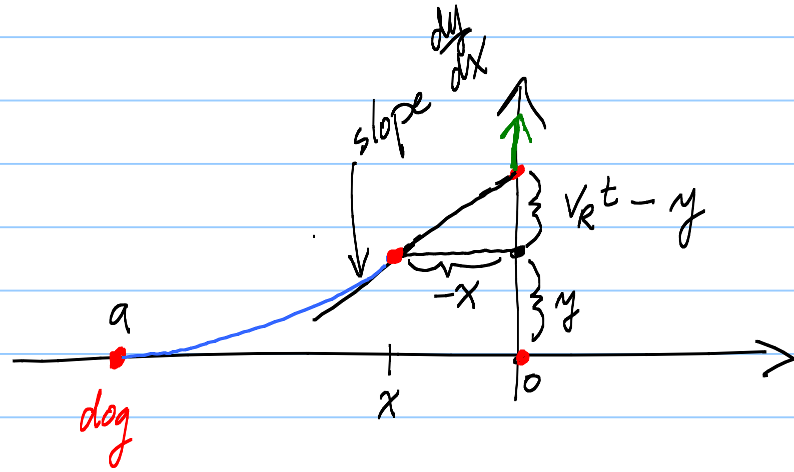
See $\lim_{t \to \infty} S(t) = 500 = (100 \text{ gal})(5 \text{ lbs/gal})$ ✓

Lake Erie. $S(0) =$ tons of mercury. fresh water in now

$S(t) \to 0$ as $t \to \infty$, but never zero.

Dog vs rabbit prob:





$a < 0$

$$\frac{dy}{dx} = \frac{V_R t - y}{(-x)}$$

Hmmm. How far has dog gone?

$$V_D t = \int_{x_0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

← solve for t
plug in above

$$\frac{dy}{dx} = \frac{\frac{V_R}{V_D} \int_{x_0}^{x} \sqrt{1+(y')^2}\, dx - y}{(-x)}$$

$$-x \frac{dy}{dx} + y = \frac{V_R}{V_D} \int_{x_0}^{x} \sqrt{1+(y')^2}\, dx$$

Aha! Diff'tiate and use fund thm calc.

$$\left(\cancel{-1 \cdot \frac{dy}{dx}} - x \frac{d^2y}{dx^2}\right) + \cancel{\frac{dy}{dx}} = \frac{V_R}{V_D} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{V_R}{V_D}\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{-x}}$$

Whoa! Let $p = \frac{dy}{dx}$ $\quad \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$$\boxed{\frac{dp}{dx} = \frac{\frac{V_R}{V_D}\sqrt{1+p^2}}{(-x)}}$$

Separable!

Take $a=-1$, $v_R = v_D$.

$$\int \frac{dp}{\sqrt{1+p^2}} = -\int \frac{1}{x}\, dx$$

$$\sinh^{-1} p = -\ln|x| + C \quad \text{Wow!}$$