Lecture 8 1.6 Exact first order ODE

HW07 due tonight in MyLab

Prob: Find an ODE solved by family curves

$$x^2 y + y^3 = C$$ ← defines fcn $y(x)$ implicity

Take $\frac{d}{dx}$ of both sides. Keep in mind that y is an unknown fcn of x. [x is x.]

$$\frac{d}{dx}(x^2 y) + \frac{d}{dx} y^3 = \frac{d}{dx} C = 0$$

$$\left(2x \cdot y + x^2 \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$\boxed{(2xy) + (x^2 + 3y^2)\frac{dy}{dx} = 0}$$

Want to go other way.

Chain rule:
$$\frac{d}{dx}\varphi(u,v) = \frac{\partial \varphi}{\partial u}\frac{du}{dx} + \frac{\partial \varphi}{\partial v}\frac{dv}{dx}$$

fcns of x

Suppose $\varphi(x,y) = C$

$u(x) = x$ $v(x) = y(x)$

$$\frac{d}{dx}\varphi(x,y) = \frac{d}{dx} C = 0$$

$$\frac{\partial \varphi}{\partial x}\underbrace{\frac{dx}{dx}} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

ODE : $\boxed{\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi}{\partial y}\dfrac{dy}{dx}}$

Big idea: Hmmm. ODE : $\underbrace{M(x,y)}_{} + \underbrace{N(x,y)}_{}\dfrac{dy}{dx} = 0$

Aha! Want φ with $\dfrac{\partial \varphi}{\partial x}$ $\dfrac{\partial \varphi}{\partial y}$

Need $\boxed{\dfrac{\partial \varphi}{\partial x} = M, \quad \dfrac{\partial \varphi}{\partial y} = N}$

Famous problem in Physics: When is a force field

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

conservative (work indep of path)?

Ans: 1) There is a potential fcn φ for $\vec{F}$, meaning that $\boxed{\nabla \varphi = \vec{F}}$

$$\int_{\gamma_A^B} \vec{F} \cdot d\vec{s} = \int_{\gamma_A^B} \nabla \varphi \cdot d\vec{s} = \varphi(B) - \varphi(A)$$

2) φ exists exactly when

$$\text{Curl } \vec{F} = 0 \qquad (\nabla \times \vec{F})$$

(no holes in space).

$$\text{Curl } \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Hmmm: $\vec{F} = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath} + 0\cdot\hat{k}$

$$\operatorname{Curl}\vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{bmatrix}$$

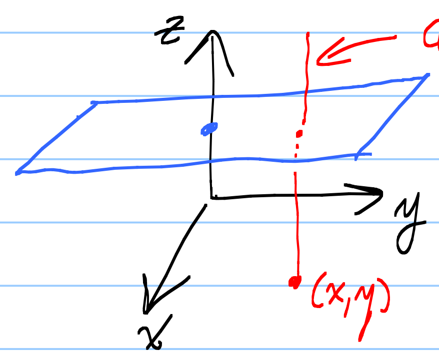
$$= \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{=0}\hat{k}$$

$= 0$: Exactness criterion.

Suppose $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Get φ with $\nabla\varphi = \underbrace{M}_{\frac{\partial\varphi}{\partial x}}\hat{\imath} + \underbrace{N}_{\frac{\partial\varphi}{\partial y}}\hat{\jmath} + \underbrace{0}_{\frac{\partial\varphi}{\partial z}}\cdot\hat{k}$

Aha! $\frac{\partial\varphi}{\partial z} \equiv 0$ means φ does not depend on z !



$\varphi(x,y,z)$ is const on | because $\frac{\partial\varphi}{\partial z} \equiv 0$

$\varphi(x,y,z) = \varphi(x,y)$

Get $\varphi(x,y)$ with $\frac{\partial\varphi}{\partial x} = M$, $\frac{\partial\varphi}{\partial y} = N$

and now $\varphi(x,y) = C$ defines

sol^{n}s to orig prob $M + N\frac{dy}{dx} = 0$!

Summary: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

If $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$, then we can cook up φ

with $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$ and

$\varphi(x,y) = C$ define the sol'ns to ODE.

EASY direction: If $\underbrace{\frac{\partial \varphi}{\partial x} = M}_{\text{Take } \frac{\partial}{\partial y}}$ and $\underbrace{\frac{\partial \varphi}{\partial y} = N}_{\text{Take } \frac{\partial}{\partial x}}$

$$\underbrace{\frac{\partial^2 \varphi}{\partial y \partial x}} = \frac{\partial M}{\partial y} \qquad \underbrace{\frac{\partial^2 \varphi}{\partial x \partial y}} = \frac{\partial N}{\partial x}$$

mixed partials =!

So $\boxed{M_y = N_x}$ ✓

Other direction MA 261.

Notation and how to remember: $M + N\frac{dy}{dx} = 0$

Multiply by dx: $\underbrace{M\,dx + N\,dy}_{\text{differential 1-form}} = 0$

$M\,dx + N\,dy$ is exact if $= d\varphi = \underbrace{\frac{\partial \varphi}{\partial x}}_{M} dx + \underbrace{\frac{\partial \varphi}{\partial y}}_{N} dy$

How to remember the exactness criterion:

Need $\frac{\partial}{\partial y}\Big[\underbrace{\text{Thing by } dx}_{M}\Big] = \frac{\partial}{\partial x}\Big[\underbrace{\text{Thing by } dy}_{N}\Big]$

Prob: $\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}\frac{dy}{dx} = 0$

Exact? $\frac{\partial}{\partial y} M = 2y + 0$

$\frac{\partial}{\partial x} N = 2y + 0$

$=$ ✓ Exact.

Can find φ with $\begin{cases} \frac{\partial \varphi}{\partial x} = M = y^2 + \cos x & (A) \\ \frac{\partial \varphi}{\partial y} = N = 2xy + \sin y & (B) \end{cases}$

Step 1: First use (A):

$$\varphi = \int y^2 + \cos x \, dx$$

Integrate, treating y like a const.

$$\varphi = y^2 x + \sin x + C(y)$$

Unknown fcn of y

Why $C(y)$ instead of just C? Answer: If φ_1 and φ_2 have same $\frac{\partial}{\partial x}$: $\frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_2}{\partial x}$ $\therefore$ $\frac{\partial}{\partial x}(\varphi_1 - \varphi_2) = 0$

$\frac{\partial}{\partial x}(\varphi_1 - \varphi_2) \equiv 0$

$\varphi_1 - \varphi_2$ const along horiz lines.

So $\varphi_1 - \varphi_2 =$ fcn of y alone.

(A) is used up. Got

$$\boxed{\varphi(x,y) = xy^2 + \sin x + C(y)} \quad (A)$$

Next use (B) (in a different way).

$$(B): \quad \frac{\partial \varphi}{\partial y} = \underbrace{2xy^2 + \sin y}_{N} \quad \text{want}$$

$$\frac{\partial}{\partial y}\Big[\underbrace{xy^2 + \sin x + C(y)}_{\varphi \text{ from } (A)}\Big] \overset{\text{want}}{=} 2xy + \sin y$$

$$\cancel{2xy} + 0 + C'(y) = \cancel{2xy} + \sin y$$

Miracle! (Exactness criterion makes all x's cancel.)

$$\boxed{C'(y) = \sin y}$$

$$C(y) = \int \sin y \, dy = -\cos y$$

(Skip $+C$ here.)

Finally, $\varphi(x,y) = xy^2 + \sin x \underbrace{- \cos y}_{C(y)} = C$

Last step: Set $\varphi(x,y) = \text{const.}$

Solve for y if you can.