Lecture 9   2.1 Population dynamics

HW08 due MyLab tonight

EXACT Equ ex:

$$\underbrace{(ye^{xy}+2x)}_{M}+\underbrace{(xe^{xy}-y)}_{N}\frac{dy}{dx}=0$$

Exactness test: $M_y \overset{?}{=} N_x$

$$\begin{cases} \dfrac{\partial M}{\partial y} = 1\cdot e^{xy} + y(xe^{xy}) + 0 \\ \dfrac{\partial N}{\partial x} = 1\cdot e^{xy} + x(ye^{xy}) + 0 \end{cases} = \text{Exact} \checkmark$$

$y''+y=0$

x missing

$p=\frac{dy}{dx}$

$y''=\frac{dp}{dx}=\underbrace{\frac{dp}{dy}\frac{dy}{dx}}_{p\frac{dp}{dy}}$

So φ exists such that

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = M = ye^{xy}+2x & (A) \\ \dfrac{\partial\varphi}{\partial y} = N = xe^{xy}-y & (B) \end{cases}$$

Step 1: Use (A):

$$\varphi = \int ye^{xy}+2x\,dx$$

treat y like a constant

$$= \underbrace{y\left(\frac{1}{y}e^{xy}\right)+x^2+\underbrace{C(y)}_{\text{arb fcn } y}}_{\varphi \text{ from (A)}}$$

Step 2: Use (B) (differently!)

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\Big(\underbrace{e^{xy}+x^2+C(y)}_{\varphi \text{ from (A)}}\Big) \overset{\text{want}}{=} N = xe^{xy}-y$$

$$\cancel{e^{xy}(x)}+0+C'(y) = \cancel{xe^{xy}}-y$$

$$\boxed{C'(y) = -y}$$

$$C(y) = \int -y\,dy = -\frac{1}{2}y^2 + C_1$$

can skip

Finally $\quad \varphi = e^{xy} + x^2 + \underbrace{\left(-\frac{1}{2}y^2 + C_1\right)}_{C(y)} = C$

can absorb!

important

Ans: $\quad e^{xy} + x^2 - \frac{1}{2}y^2 = C$

Get C from initial conditions. Solve for y if you can.

Population dynamics: $\quad \frac{dp}{dt} = kp \quad$ Malthusian

Logistic eqn: $\quad \frac{dp}{dt} = [k(M-p)]\,p \quad$ ← Separable! Use partial fractions.

$$\frac{dp}{dt} \underbrace{-kM}_{P(t)} p = \underbrace{-k}_{Q(t)} p^{2} \quad n=2$$

← Bernoulli eqn

Let $\quad v = p^{1-n} = p^{1-2} = p^{-1} = \frac{1}{p}$

Get new eqn: $\quad \boxed{\frac{dv}{dt} + (1-n)P(t)\,v = (1-n)\,Q(t)}$

$$\frac{dv}{dt} + (1-2)(-kM)\,v = (1-2)(-k)$$

$$\boxed{\frac{dv}{dt} + kM\,v = k} \quad \text{Linear!}$$

How? Why? $\quad v = \frac{1}{p} \quad \frac{dv}{dt} = (-1)p^{-2}\frac{dp}{dt} = \underline{-\frac{1}{p^2}\frac{dp}{dt}}$

$$\left[\frac{dp}{dt} - kMp = -kp^2\right] \cdot \left(-\frac{1}{p^2}\right)$$

$$\underline{-\frac{1}{p^2}\frac{dp}{dt}} - kM\frac{1}{p} = -k$$

$$\frac{dv}{dt} - kMv = -k \quad \checkmark$$

1. $\frac{dv}{dt} + kMv = k$

Int. factor : $u = e^{\int kM\,dt} = e^{kMt}$

$$\underbrace{e^{kMt}\left(\frac{dv}{dt} + kMv\right)}_{\left[e^{kMt}v\right]'} = ke^{kMt}$$

$$e^{kMt}v = \int ke^{kMt}\,dt = \frac{k}{kM}e^{kMt} + C$$

$$\boxed{v = \frac{1}{M} + Ce^{-kMt}}$$

$v = \frac{1}{p}$

Finally $\quad p = \dfrac{1}{\frac{1}{M} + Ce^{-kMt}}$

What is C? Initial condition: $p(0) = p_0$

Plug in $t=0$: $\underbrace{P(0)}_{p_0} = \dfrac{1}{\frac{1}{M} + Ce^0} = \dfrac{1}{\frac{1}{M} + C}$

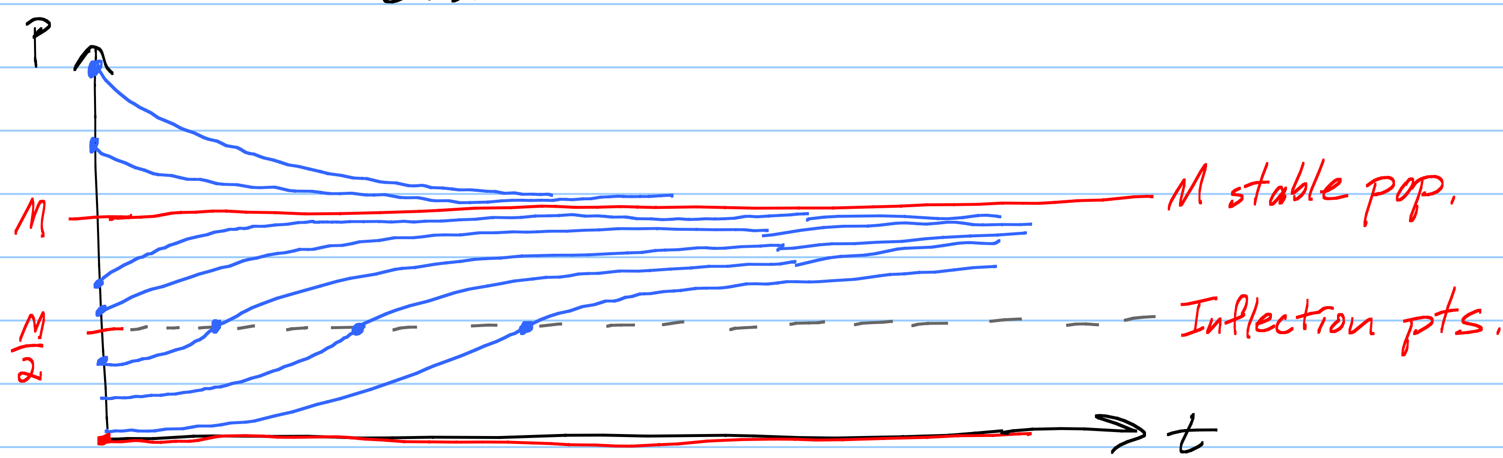
$$\frac{1}{p_0} = \frac{1}{M} + C$$

$$\boxed{C = \frac{1}{p_0} - \frac{1}{M}}$$

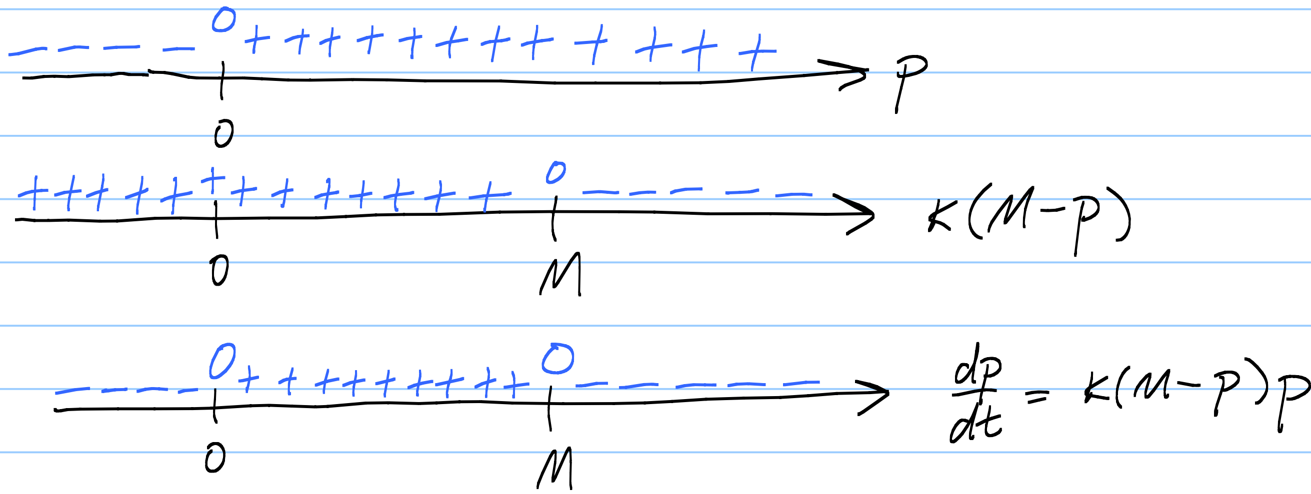
So
$$P = \frac{1}{\frac{1}{M} + \underbrace{\left(\frac{1}{p_0} - \frac{1}{M}\right)}_{C} e^{-kMt}} \cdot \frac{p_0 M}{p_0 M}$$

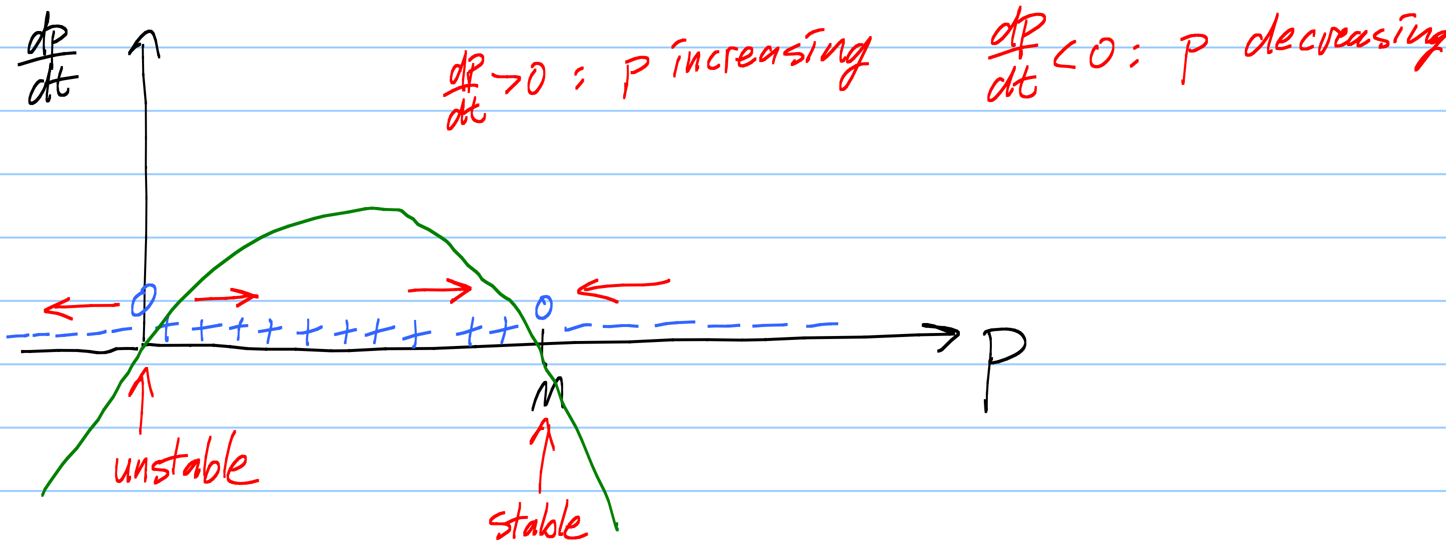
$$\boxed{P(t) = \frac{p_0 M}{p_0 + (M - p_0) e^{-kMt}}}$$

See
$$\lim_{t\to\infty} p(t) = \frac{p_0 M}{p_0 + 0} = M$$



$$\frac{dp}{dt} = k(M-p)p \qquad \text{Sign of } \frac{dp}{dt}$$





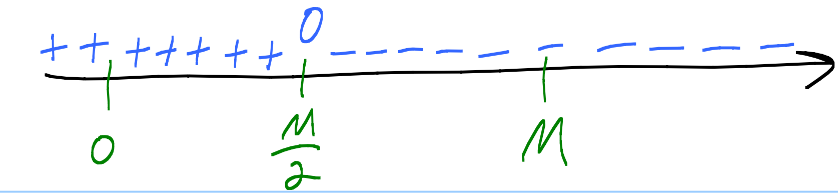
Inflection points?

$$\frac{d^2p}{dt^2} = \frac{d}{dt}\left(\frac{dp}{dt}\right)$$

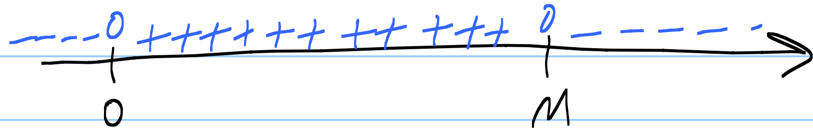
$$= \frac{d}{dt}\left(k(M-p)p\right) = \frac{d}{dt}\left[-kMp - kp^2\right]$$

$$= -kM\frac{dp}{dt} - k\,2p\,\frac{dp}{dt}$$

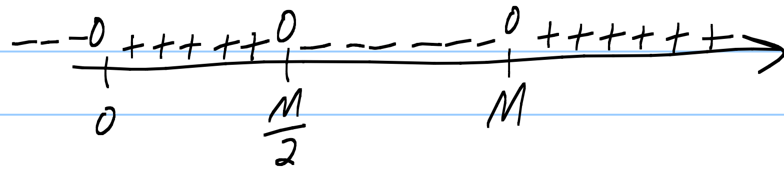
$$= \underbrace{\frac{dp}{dt}}_{\text{know sign}}\left(-kM - 2kp\right)$$



$-kM-2kP$



$\frac{dP}{dt}$



sign $\frac{d^2P}{dt^2}$

↑ inflection points when cross $P = \frac{M}{2}$.