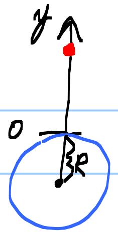
Lecture 12   2.4, 2.5 Euler's method

HW11 due in MyLab tonight



$$F = ma$$

$$-\frac{GMm}{(y+R)^2} = m\frac{dv}{dt}$$

$$= \frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v} = v\frac{dv}{dy}$$

Got
$$v^2 = \frac{2GM}{y+R} \underbrace{- \frac{2GM}{R} + v_0^2}_{+C}$$

$$\int \frac{dy}{\sqrt{\frac{2GM}{y+R} + C}} = \int dt = t + K$$

Case: $v_0 = \sqrt{\frac{2GM}{R}}$ ← v_e escape velocity. $C = 0$

$$\int \sqrt{y+R}\, dy = \int \sqrt{2GM}\, dt$$

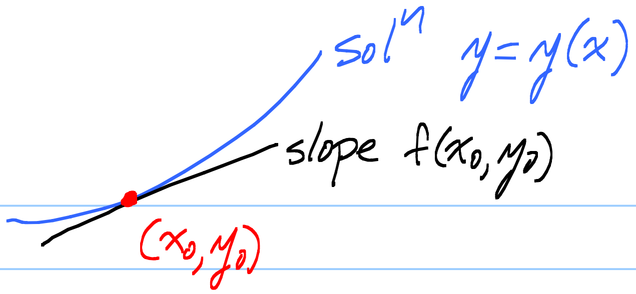
$$\frac{1}{\frac{1}{2}+1}(y+R)^{\frac{1}{2}+1} = \sqrt{2GM}\, t + K$$

$t=0$; $y=0$. $\boxed{K = \frac{2}{3}R^{3/2}}$

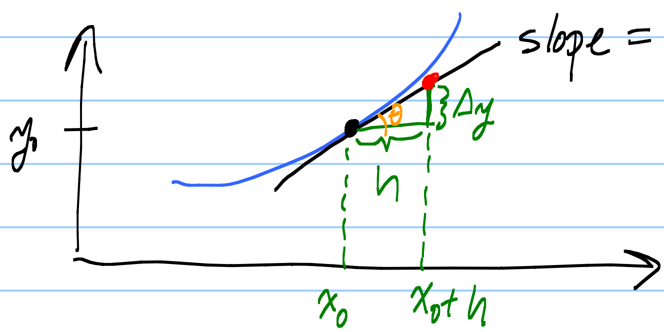
Solve for $y = \left(\frac{3}{2}\sqrt{2GM}\, t + R^{3/2}\right)^{2/3} - R$

Euler's method:

$$\frac{dy}{dx} = f(x,y), \quad f(x_0) = y_0$$



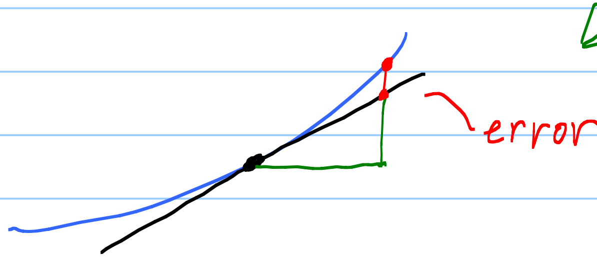
Big idea: Follow tangent line for small $\Delta x = h$



slope $= f(x_0,y_0) = \tan\theta$

$$\Delta y = h \tan\theta = h f(x_0,y_0)$$

red dot: $\left((x_0+h),\ y_0 + \underbrace{h f(x_0,y_0)}_{\Delta y}\right)$



Euler method:

$$\begin{array}{ll} x_0 & y_0 \\ x_1 = x_0 + h & y_1 = y_0 + h f(x_0,y_0) \\ x_2 = x_1 + h & y_2 = y_1 + h f(x_1,y_1) \\ x_3 = x_2 + h & y_3 = y_2 + h f(x_2,y_2) \\ \vdots & \vdots \end{array}$$

$$\boxed{x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h f(x_n,y_n)}$$

EX: $\dfrac{dy}{dx} = \underbrace{1 - x + 4y}_{f(x,y)}$ $\qquad y(0) = 1$ (Linear!)

$x_0 = 0 \qquad y_0 = 1$

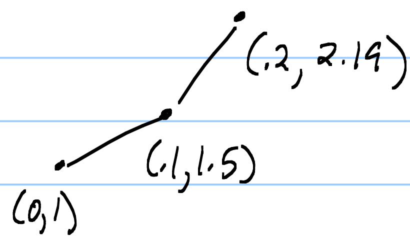
Step size: $h = .1$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= (1) + (.1)\left[1 - 0 + 4 \cdot 1\right] = 1.5$$

$$\boxed{(x_1, y_1) = (.1, 1.5)}$$

$x_2 = .2$

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= (1.5) + (.1)\left[1 - (.1) + 4(1.5)\right] = 2.19$$



Excel sheet!

n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + h f(x_n, y_n)$
0	0	1		1.5
1	.1	1.5		
2	.2			
3	.3			

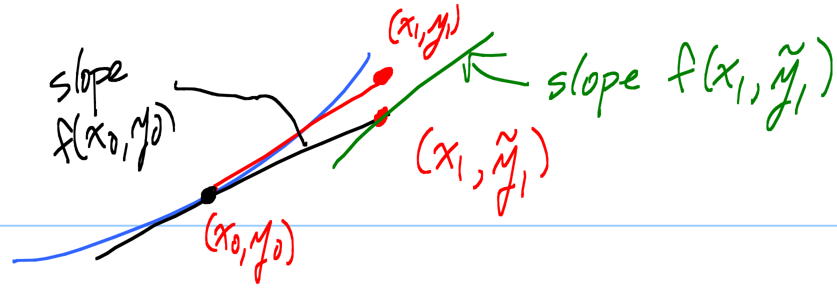
Easy case: $\dfrac{dy}{dx} = f(x)$

Euler method: $$y_N = y_0 + \sum_{k=0}^{N-1} f(x_k)\, h$$ Riemann sum!

$$\longrightarrow y_0 + \int_{x_0}^{x_N} f(t)\, dt$$

Actual sol^n!

Improved Euler method:



Big idea: Improve slope guess by averaging these two

$$\tilde{y}_1 = \underbrace{y_0 + h f(x_0, y_0)}$$

Slope 1: $f(x_0, y_0)$ Slope 2: $f(x_1, \tilde{y}_1)$

Improved slope: $\dfrac{f(x_0,y_0) + f(x_1, y_0 + h f(x_0,y_0))}{2}$

$$y_1 = y_0 + h\left[\frac{f(x_0,y_0) + f(x_1, y_0 + h f(x_0,y_0))}{2}\right]$$

Summary:

$$x_{n+1} = x_n + h$$

$$\tilde{y}_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2}\right]$$

Cool thing: $\dfrac{dy}{dx} = f(x)$. Imp. Eul ~ Trapezoid rule

Runge-Kutta ~ Simpson's rule!