Lecture 13 — 3.1 2nd order linear ODE

HW12 due tonight in MyLab
HW09w, HW10w, HW11w in Gradescope

Euler method failing: $\frac{dy}{dx} = y^2 \leftarrow$ sol'ns blow up in finite time.
Euler doesn't even notice!
Sol'ns appear to go on and on, and get weird as h shrinks.

Easiest 2nd order ODE:

$$\frac{d^2y}{dx^2} = -\sin x$$

$$\frac{dy}{dx} = -\int \sin x \, dx = -\cos x + C_1$$

$$y = \int (-\cos x + C_1)\, dx + C_2$$

Sol'n:

$$y = \underbrace{\sin x}_{\substack{\text{particular} \\ \text{sol'n} \\ y_p}} + \underbrace{C_1 x + C_2}_{\text{Gen'l sol'n to } y''=0}$$

$y''=0 \uparrow$ homog eqn

2nd order linear ODE:

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)\,y = D(x)$$

Standard Form: Divide by $A(x)$:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \qquad (*)$$

$\uparrow$ 1 here.

$$P = \frac{B}{A}, \quad Q = \frac{C}{A}, \quad R = \frac{D(x)}{A(x)}$$

zeroes of $A(x)$ dangerous!

E. & U. Thm: $(*)$ plus two initial conditions

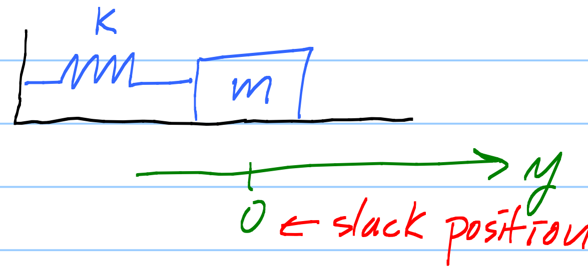
$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$ has a unique sol^n on any interval

has ↑ sol^n exists (E.)

unique ↑ (U.)

where $P(x)$, $Q(x)$, and $R(x)$ are all <u>continuous</u>.

<u>EX</u>: $y'' + 3y' + 2y = 0$



$F = ma$

$$\underbrace{-ky}_{\text{Hooke's law}} \; \underbrace{-cy'}_{\text{friction}} = my''$$

Could try "Reduction of order" because x is missing:

$v\frac{dv}{dy} + 3v + 2y = 0$ Ugh!

Hmmm. Try $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

Plug in: $y'' + 3y' + 2y = 0$

$(r^2 e^{rx}) + 3(re^{rx}) + 2e^{rx} \overset{\text{want}}{=} 0$

$$\underbrace{\left[r^2 + 3r + 2\right]}_{\text{need } r \text{ to a root of}} e^{rx} \equiv 0$$

e^{rx} <u>never</u> zero

$$\boxed{r^2 + 3r + 2 = 0}$$ ← "Characteristic equation"

$$(r+2)(r+1)=0$$

Get <u>two</u> sol^{n}s : $r=-1,-2$ $\quad y_1 = e^{-x}$

$y_2 = e^{-2x}$

Important property of linear homogeneous $(R(x)\equiv 0)$ equs

Superposition principle: If y_1 and y_2 solve the linear homog ODE, then <u>so does</u> $c_1 y_1 + c_2 y_2$.

<u>Why</u>: Write $L[y] = y'' + Py' + Qy$

L is a <u>linear</u> operator meaning that

$$L[\underbrace{c_1y_1+c_2y_2}_{\text{a sol}^n\text{ too!}}] = c_1 \underbrace{L[y_1]}_{=0,\ y_1\text{ sol}^n} + c_2 \underbrace{L[y_2]}_{=0,\ y_2\text{ sol}^n} = 0$$

Check: $L[c_1y_1+c_2y_2] = (c_1y_1+c_2y_2)'' + P(c_1y_1+c_2y_2)' + Q(c_1y_1+c_2y_2)$

$$= c_1y_1'' + c_2y_2'' + c_1Py_1' + c_2Py_2' + c_1Qy_1 + c_2Qy_2$$

$$= c_1L[y_1] + c_2L[y_2] \quad \checkmark$$

Aha! $y = c_1e^{-x} + c_2e^{-2x}$ solves prob!

Big question: Is it the gen^{l} sol^{n}?

Equiv quest: Can I solve <u>any</u> and <u>all</u> initial value probs (IVP)?

Hmmm: $y = c_1 e^{-x} + c_2 e^{-2x}$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

Can I find c_1 and c_2 so that

want

$$\begin{cases} y(0) = c_1 e^{-0} + c_2 e^{-2\cdot 0} = A \\ y'(0) = -c_1 e^{-0} - 2c_2 e^{-2\cdot 0} = B \end{cases}$$

any old A and B

$$\begin{cases} c_1 + c_2 = A \\ -c_1 - 2c_2 = B \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Cramer's rule:

$$c_1 = \frac{\det \begin{bmatrix} A & 1 \\ B & -2 \end{bmatrix}}{\underbrace{\det \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{1\cdot(-2) - 1(-1) = -1}} = \frac{-2A - B}{-1}$$

not zero!

$$c_2 = \frac{\det \begin{bmatrix} 1 & A \\ -1 & B \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}} = \frac{B + A}{-1}$$

Because det in denom is $\neq 0$, can solve any and all IVPs with $c_1 y_1 + c_2 y_2$! Conclude $c_1 y_1 + c_2 y_2$ is the <u>gen^{l} sol^{n}</u>.

General problem: Two sol^{n}s y_1, y_2 to homog prob.

$$y = c_1 y_1 + c_2 y_2$$
$$y' = c_1 y_1' + c_2 y_2'$$

want

$$\begin{cases} y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = A \\ y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = B \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\text{need } \det \neq 0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Def^n: Wronskian of y_1 and y_2 at x_0 is

$$W = W(x_0) = W[y_1, y_2](x_0) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

EX: $W[e^{-x}, e^{-2x}](x)$

$$= \det \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} = -2e^{-3x} - \left(-e^{-3x}\right)$$
$$= e^{-3x} \leftarrow \text{never zero!}$$

<u>Good news</u>: Can solve any and all IVPs at any x_0 with $c_1 y_1 + c_2 y_2$.

<u>Bummer</u>: $L[y] = ay'' + by' + cy = 0$

$$L[e^{rx}] = ar^2 e^{rx} + bre^{rx} + ce^{rx} \overset{\text{want}}{=} 0$$

$$\underbrace{(ar^2 + br + c)}_{\text{need } = 0} e^{rx} = 0$$

Roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Q! What if $b^2 - 4ac = 0$? Only get <u>one</u> sol^n! Don't get gen^l sol^n.

What if $b^2 - 4ac < 0$? Complex roots!