Lecture 14 3.2 2nd order linear ODE with constant coeff

HW13 due MyLab tonight

Last time: Find two sol'ns y_1 & y_2

So that $W(x_0) \neq 0$

$$\det\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \neq 0$$

Then $c_1 y_1 + c_2 y_2$ is the gen'l sol'n (can solve all IVPs).

Cool fact: Abel's formula shows that either

$$W \equiv 0$$

or W is never zero.

Only need to test W at one pt.

homog eq'n

Abel: Say y_1, y_2 solve $y'' + Py' + Qy = 0$

Then

$$\left(y_2'' + Py_2' + Qy_2 = 0\right) \cdot y_1$$

$$\left(y_1'' + Py_1' + Qy_1 = 0\right) \cdot y_2 \quad -$$

$$\underbrace{(y_1 y_2'' - y_2 y_1'')}_{\frac{dW}{dx}} + P\underbrace{(y_1 y_2' - y_2 y_1')}_{W} = 0$$

Aha!

Abel's: $\frac{dW}{dx} + PW = 0$ Separable!

$$\int \frac{dW}{W} = -\int P\,dx$$

$$\ln|W| = -\int P\,dx + C$$

$$|W| = e^{C} e^{-\int P\,dx}$$

$$W = K \underbrace{e^{-\int P\,dx}}_{\text{never } 0} \qquad \boxed{K = \pm e^{C}}$$

Consequently, $W \equiv 0$ $(K=0)$ or never zero $(K \neq 0)$.

Meaning of $\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = 0$:

Think about system
$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = A = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = B = y_0' \end{cases}$$

Det $\neq 0$ meant we could use Cramer's rule to get c's.

What if Det $= 0$?

Hmmm.

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{cases}$$

If $\det \neq 0$, Cramer's rule yields $c_1 = 0$, $c_2 = 0$.

Fact: When Cramer's rule fails, i.e., when $\det = 0$, ther are non-zero vector sol^n's $\binom{c_1}{c_2}$ solving homog system. Say $\binom{a_1}{a_2}$ is one such vector $\neq \vec{0}$.

$$\begin{cases} a_1 y_1(x_0) + a_2 y_2(x_0) = 0 \\ a_1 y_1'(x_0) + a_2 y_2'(x_0) = 0 \end{cases}$$

Aha! $a_1 y_1 + a_2 y_2$ solves ODE with $y_0 = 0$, $y_0' = 0$ initial conds! So does $y \equiv 0$ sol^n!

Uniques of E & U Thm $\Rightarrow$ $a_1 y_1 + a_2 y_2 \equiv 0$.

Say $a_2 \neq 0$. Then we see $\boxed{y_2 = \frac{a_1}{a_2} y_1}$

Fact: If $W = 0$, then one of the y's $=$ (constant) times the other! They are linearly dependent.

Consequence: Get y_1. Only need to find a sol^n y_2 that is "different than y_1" (meaning $y_2 \neq$ (const) y_1.) Then $W \neq 0$. Get gen^l sol^n.

Last time: $ay'' + by' + cy = 0$ (const coeff).

Try $y = e^{rx}$: $\underbrace{(ar^2 + br + c)}_{=0} e^{rx} = 0$ ← want

Case: Two real roots r_1, r_2 with $r_1 \neq r_2$.

Then gen^l sol^n is $\boxed{c_1 e^{r_1 x} + c_2 e^{r_2 x}}$

↖ best case!

Second best case:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \sqrt{b^2-4ac} = 0$$

Only one root: $r_1 = \frac{-b}{2a}$

Get $y_1 = e^{r_1 x}$ only. Need y_2.

Ex: $y'' + 4y' + 4y = 0$

Charac. eqn: $\boxed{r^2 + 4r + 4 = 0}$

$$(r+2)^2 = 0$$

Repeated roots: $r = -2, -2$ Get $y_1 = e^{-2x}$.

Euler: $L[y] = y'' + 4y' + 4y$

$$L[e^{rx}] = (r^2 + 4r + 4)e^{rx} = (r+2)^2 e^{rx}$$

Take $\frac{\partial}{\partial r}$ of this eqn!

$$\underbrace{\frac{\partial}{\partial r} L[e^{rx}]}_{L\left[\underbrace{\frac{\partial}{\partial r} e^{rx}}_{xe^{rx}}\right]} = \frac{\partial}{\partial r}\left[(r+2)^2 e^{rx}\right]$$

$$= \underline{2(r+2)e^{rx} + (r+2)^2 x e^{rx}}_{\;r=-2}$$

because L is linear

Euler Aha! Now take $r = -2$, RHS $= 0$

$$L[\underbrace{xe^{-2x}}_{y_2}] \equiv 0$$

Joy! Got $y_2 = xe^{-2x}$, which is not a const $\times\, y_1$.

Repeated roots case: Get one root r_1, r_1; gen' sol'n is

$$\boxed{c_1 e^{r_1 x} + c_2 x e^{r_1 x}}$$

Best case: $b^2 - 4ac > 0$.
Repeated root: $b^2 - 4ac = 0$
complex roots!: $b^2 - 4ac < 0$

Euler: pretend not to notice!

$$e^{a+bi} = 1 + (a+bi) + \frac{(a+bi)^2}{2!} + \frac{(a+bi)^3}{3!} + \cdots$$

$$= e^{a} e^{bi} \qquad \text{(binomial thm)}$$

$$= e^{a}\Big[\underbrace{1 + bi + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \cdots}_{1 + bi - \frac{b^2}{2!} - i\frac{b^3}{3!} + \frac{b^4}{4!} + \frac{ib^5}{5!} - \cdots}\Big]$$

$$= e^{a}\Big[\underbrace{\Big(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \cdots\Big)}_{\text{Cos } b} + i\underbrace{\Big(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \cdots\Big)}_{\text{Sin } bx}\Big]$$

Euler's formulas: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\underbrace{e^{a+bi} = e^{a}e^{bi} = e^{a}(\cos b + i \sin b)}_{\text{Def}^n \text{ of complex expontential}}$$

<u>Euler</u>: Hmmm. Get complex root $r = a+bi$

Get complex sol^n: $y_1 = e^{rx} = e^{(a+bi)x}$

$$= e^{ax} e^{ibx}$$

$$= e^{ax}(\cos bx + i \sin bx)$$

$$= \underbrace{e^{ax}\cos bx}_{u_1} + i\,\underbrace{e^{ax}\sin bx}_{u_2}$$

This <u>is</u> a complex sol^n $y = u_1(x) + i\,u_2(x)$.

<u>Fact</u>: The $\underbrace{\text{real part}}_{u_1}$ and $\underbrace{\text{imaginary part}}_{u_2}$ are

also sol^{n}s. One complex sol^n yields <u>two</u> real sol^{n}s.

<u>Complex root case</u>: $r = a \pm bi$

Do this using $r = a + bi$. Get gen^l sol^n

$$\boxed{y = c_1 \underbrace{e^{ax}\cos bx}_{y_1} + c_2 \underbrace{e^{ax}\sin bx}_{y_2}}$$

(What if we use $r = a - bi$ instead of $a + bi$.

Get same thing! $\cos(-bx) = \cos bx$

$\sin(-bx) = -\sin bx$ ✓