Lecture 15

3.3 2^{nd} order linear ODE with constant coeff and higher order linear eqns

HW14 due MyLab tonight

Last time $ay'' + by' + cy = 0$ homogeneous

Char eqn: $ar^2 + br + c = 0$

Roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Case 1: r_1, r_2 real and unequal $(b^2 - 4ac > 0)$

Gen^l sol^n $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: $r_1 = r_2$ (real) $(b^2 - 4ac = 0)$

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Case 3: $r_1, r_2 = A \pm Bi$ complex roots $(b^2 - 4ac < 0)$

$$y = c_1 e^{Ax} \cos Bx + c_2 e^{Ax} \sin Bx$$

Complex solutions: $\Psi(x) = u(x) + i\, v(x)$

$$\frac{\Psi(x+h) - \Psi(x)}{h} = \left(DQ\, u\right) + i\left(DQ\, v\right)$$

$$\longrightarrow u'(x) + i\, v'(x)$$

Key fact: Complex sol^n yields two real sol^{n}s:

$$0 = L[\Psi] = a\Psi'' + b\Psi' + c\Psi$$

$$= \underline{a}(u'' + i\underline{v''}) + \underline{b}(u' + i\underline{v'}) + \underline{c}(u + i\underline{v})$$

$$= \underbrace{L[u]}_{=0} + i\,\underbrace{L[v]}_{=0}$$

✓

Why it works: $\frac{d}{dx} e^{(A+Bi)x} = (A+Bi)\,e^{(A+Bi)x}$ ← check this.

$$L[e^{rx}] = \underbrace{(ar^2+br+c)}\, e^{rx} = 0$$

even if r is a complex root

Wronskian of:

$$\det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ Ae^{Ax}\cos Bx - Be^{Ax}\sin Bx & Ae^{Ax}\sin Bx + Be^{Ax}\cos Bx \end{bmatrix}$$

$$= \det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ -Be^{Ax}\sin Bx & Be^{Ax}\cos Bx \end{bmatrix} \leftarrow (\text{Row } 2) - A\cdot(\text{Row } 1)$$

$$= Be^{Ax}\underbrace{(\cos^2 Bx + \sin^2 Bx)}_{=1} = Be^{Ax} \leftarrow \text{never zero}$$

Great! They form the gen'l sol'n!

Remark: Did I really have to compute W?

No! All I need to check is $\left(\frac{y_1}{y_2}\right) = \text{Const.}$

$$\frac{y_1}{y_2} = \frac{e^{Ax}\cos Bx}{e^{Ax}\sin Bx} = \frac{\cos}{\sin} = \cot Bx \leftarrow \text{not a constant!}$$

So y_1 and y_2 are linearly independent and therefore form the gen$^{\ell}$ sol^{n}.

Higher order eqns:

Ex: $ay''' + by'' + cy' + dy \overset{\text{homog}}{=} 0$

IVP: $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ y''(x_0) = y_0'' \end{cases}$

Think: Need $\underline{\underline{3}}$ conditions to pin down $\underline{\underline{3}}$ arb. constants.

Play same game. Try e^{rx}. Get cubic char eqn.

Get 3 roots. Get $y = c_1y_1 + c_2y_2 + c_3y_3$

IVP: Need
$$\begin{cases} y(x_0) = c_1y_1(x_0) + c_2y_2(x_0) + c_3y_3(x_0) \overset{\text{want}}{=} y_0 \\ y'(x_0) = c_1y_1'(x_0) + c_2y_2'(x_0) + c_3y_3'(x_0) = y_0' \\ y''(x_0) = c_1y_1''(x_0) + c_2y_2''(x_0) + c_3y_3''(x_0) = y_0'' \end{cases}$$

Wronskian!

$$\det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

Facts: 1) If $W(x_0) \neq 0$, we get the gen$^{\ell}$ sol^{n}.

2) Abel's formula holds: Consequence: W is either $\equiv 0$ or never zero.

3) (2) $\Longrightarrow$ just need to check W at one point.

Feeling: Need to find 3 "different" sol^n's to get gen^l sol^n.

Different enough: Linearly independent

Not different enough: Linearly dependent

Def^n: y_1, y_2, y_3 are linearly independent on (a,b) if the only way to get

$$c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0 \quad \text{on } (a,b)$$

is by taking $c_1 = c_2 = c_3 = 0$.

EX: Are x^2+x+1, x^2+x, x^2 lin ind on $(-\infty,\infty)$?

Hmmm: $c_1(x^2+x+1) + c_2(x^2+x) + c_3 x^2 \equiv 0$ want

$$\underbrace{(c_1+c_2+c_3)}_{\text{must be } 0} x^2 + \underbrace{(c_1+c_2)}_{0} x + \underbrace{(c_1)}_{0} \equiv 0$$

Linear eqns

$$\begin{cases} c_1+c_2+c_3 = 0 & 3.\ c_3 = 0 \\ c_1+c_2 = 0 & 2.\ c_2 = 0 \\ c_1 = 0 & 1.\ c_1 = 0 \end{cases}$$

All c's must be 0. So y_1, y_2, y_3 are lin. ind.

EX: What about $\sin^2 x$, $\cos 2x$, 3 ?

Hummm: Know $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

Dependency relationship:

$$\sin^2 x + \frac{1}{2}\cos 2x - \frac{1}{6}(3) \equiv 0$$

$c_1 = 1$ $\quad c_2 = \frac{1}{2}$ $\quad c_3 = -\frac{1}{6}$ not all zero

So one fcn can be written as a linear combo of the others. It doesn't "add to the list".

Fact: $y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x)\, y' + P_0(x)\, y = 0$

where P's are continuous fcns on (a, b).

IVP :
$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ \quad \vdots \\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{cases}$$
n cond's

If $y_1, y_2, \ldots, y_n$ are n lin ind sol^{n}s, then

$c_1 y_1 + \cdots + c_n y_n$ is gen^{l} sol^{n}.

Equivalently : $W \neq 0$ at some pt ($\Longleftrightarrow$ never zero)

Const coeff case: factor char poly of order 10 ODE

$$r\underbrace{(r-2)^3}\underbrace{(r^2+2r+2)^2}\underbrace{(r^2+1)}=0$$

$\boxed{r=0}$

$y_1 = e^{0x} = 1$

$\boxed{r=2,2,2}$

$y_2 = e^{2x}$

$y_3 = xe^{2x}$

$y_4 = x^2e^{2x}$

$\boxed{r=-1\pm i \text{ repeated!}}$

$y_5 = e^{-x}\cos x$

$y_6 = e^{-x}\sin x$

$y_7 = xe^{-x}\cos x$

$y_8 = xe^{-x}\sin x$

$\boxed{r=\pm i} = 0\pm i$

$y_9 = e^{0x}\cos x = \cos x$

$y_{10} = \sin x$