Lecture 16 3.3 (part 2) n-th order linear homogeneous ODE with constant coeff

HW15 due Mylab
HW12w, 13w, 14w due in Gradescope

Exam 1 Tuesday, Oct. 5, 8:00–9:00 pm in UC
University Church!

[Probs 1–12 of Prac. Prob

[Exam 1:
7 multiple choice,
3 regular.
10 pts each

n-th order linear const coeff:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0$$

$$a_n r^n + a_{n-1} r^{n-1} + \cdots a_1 r + a_0 = 0 \leftarrow \text{Char. equ.}$$

Fact: Polys with real coeff can be factored into terms:

1) $(r-r_0)^n$

EX: $r = (r-0)^1$

$(r-7)$

$(r-1)^{13}$

2) and quadratic polys with complex roots

$(r^2+br+c)^n$

EX: (r^2+1) $r = \pm i$

$(r^2+4)^3$

$(r^2+2r+2)^{13}$

$$\text{Char. poly} = a_n (r-r_1)^{k_1}(r-r_2)^{k_2} \cdots (r^2+b_1 r+c_1)^{m_1} \cdots$$

Sum of k's and $2m$'s = deg poly

Simple real root case: $(r-r_0)^1$

Get sol^n $y = e^{r_0 x}$

Repeated real root case:

$$(r-r_0)^N$$

Get N sol^{n}s (lin. ind.!)

$$\left\{\begin{array}{l} e^{r_0 x} \\ x e^{r_0 x} \\ x^2 e^{r_0 x} \\ \vdots \\ x^{N-1} e^{r_0 x} \end{array}\right.$$

Euler trick

$$\frac{\partial}{\partial r}\left(e^{rx}\right)\Big|_{r=r_0}$$

$$\frac{\partial^2}{\partial r^2}\left(e^{rx}\right)\Big|_{r=r_0}$$

$$\vdots$$

Complex simple roots:

(r^2+br+c) with complex roots $A \pm Bi$

Get two sol^{n}s = real and imag part of a complex sol^n $e^{(A+Bi)x}$

$$\underline{e^{Ax}\cos Bx}, \quad \underline{e^{Ax}\sin Bx} \qquad \text{(real!)}$$

[Note: OK if $A=0$. $e^{0\cdot x}=1$. $\cos Bx$, $\sin Bx$]

Repeated complex roots:

$(r^2+br+c)^N$ complex roots $A \pm Bi$

$$\left\{\begin{array}{l} e^{(A+Bi)x} \\ x e^{(A+Bi)x} \\ \vdots \\ x^{N-1} e^{(A+Bi)x} \end{array}\right. \qquad 2N \text{ sol}^n\text{s!} \left\{\begin{array}{l} e^{Ax}\cos Bx,\ e^{Ax}\sin Bx \\ xe^{Ax}\cos Bx,\ xe^{Ax}\sin Bx \\ \\ x^{N-1}e^{Ax}\cos Bx,\ x^{N-1}e^{Ax}\sin Bx \end{array}\right.$$

$$e^{(A+Bi)x} = e^{Ax}\underbrace{e^{iBx}}_{\cos Bx + i\sin Bx}$$

Factoring n-th deg polys with real coeff Ugh!

EX: $r^4 + 8r^2 + 16 = 0$ $\qquad y^{(4)} + 8y'' + 16y = 0$

$(r^2+4)^2 = 0$

roots $r = \pm 2i$ repeated!

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$$

Would need $\underline{\underline{4}}$ initial conds to pin down c's.

EX $r^6 + 5r^5 + 4r^4$

$= r^4(r^2 + 5r + 4)$

$= r^4 (r+4)(r+1)$

roots $\underbrace{0,0,0,0}, -4, -1$

$$y = c_1 \underset{\substack{\uparrow \\ e^{0x}}}{1} + c_2 \underset{\substack{\uparrow \\ xe^{0\cdot x}}}{x} + c_3 x^2 + c_4 x^3 + c_5 e^{-4x} + c_6 e^{-x}$$

Fact: If a poly $P(r)$ vanishes at r_0 $(P(r_0) = 0)$, then $(r - r_0)$ <u>divides</u> $P(r)$. Do long division.

EX: $r^3 - r^2 + r - 1 = P(r)$ $\qquad P(1) = 1 - 1 + 1 - 1 = 0$

So $r-1$ divides $P(r)$

$$
\begin{array}{r|l}
 & r^2 + 1 \\
\hline
r-1 & r^3 - r^2 + r - 1 \\
 & \underline{r^3 - r^2} \\
 & \qquad\quad r - 1 \\
 & \qquad\quad \underline{r-1} \\
 & \qquad\qquad 0 \leftarrow \text{no remainder}
\end{array}
$$

So $P(r) = (r-1)(r^2+1)$

$$y = c_1 e^x + c_2 \cos x + c_3 \sin x$$

Question: What if $P(2i) = 0$?

Hmmm. Roots occur in conjugate pairs $A \pm Bi$.

So $\pm 2i$ are roots

Aha! r^2+4 must be a quadratic factor!

Cool things: Operator notation: $D = \frac{d}{dx}$

$$y'' + 4y' + 4y = D^2 y + 4Dy + 4y$$

$$= (D^2 + 4D + 4)\, y$$

$$\overset{?}{=} (D+2)\left[(D+2)\, y\right]$$

dreaming

It works!

Great: $y'' + 4y' + 4y = (D+2)\, u = 0$

where $u = (D+2)y$

Easy: $(D+2)u = 0$

$$\frac{du}{dx} + 2u = 0$$

Int factor $e^{\int 2\,dx} = e^{2x}$

$$\left[e^{2x} u\right]' = 0$$

$$e^{2x} u = \int 0\,dx = c_1$$

So $\boxed{u = c_1 e^{-2x}}$

Next, need $(D+2)y = u = c_1 e^{-2x}$

$$\frac{dy}{dx} + 2y = c_1 e^{-2x}$$

Mult by e^{2x}:

$$\underbrace{e^{2x}\left(\frac{dy}{dx} + 2y\right)}_{\left[e^{2x} y\right]'} = c_1 e^{-2x} \cdot e^{2x} = c_1$$

$$e^{2x} y = \int c_1\,dx = c_1 x + c_2$$

So $y = c_1 x e^{-2x} + c_2 e^{-2x}$ Yes!

<u>Subtle point</u>: The Wronskian test assumes y's solve an ODE with <u>continuous</u> fcns for coeff's: $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0$

Then: equiv
1) $W \neq 0$ at one pt
2) W never zero
3) The y's are linearly independent
4) Linear combos of y's form gen'l sol'n.

What if y's don't come from an ODE prob?

[Still true that $W \neq 0$ at a pt implies that the y's are lin independent.] (Cramer's rule. easy)

<u>But</u>, if $W \equiv 0$, it does not necessarily mean that the y's are linear dependent!

<u>EX</u>: $y_1 = x^2$

$$y_2 = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$ Continuously diff'ble

$W \equiv 0$, but $c_1 y_1 + c_2 y_2 \equiv 0$

[Let $x = \pm 1$] $\Longrightarrow c_1 = 0$ and $c_2 = 0$.