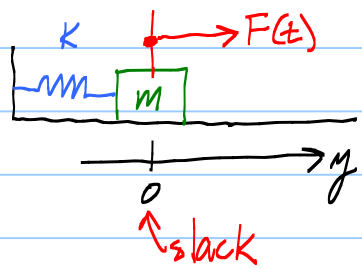
Lecture 17 3.4 Vibrations

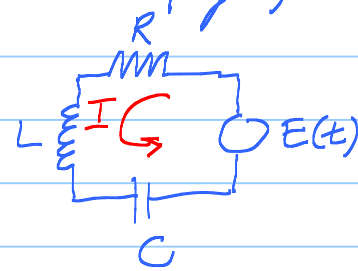
HW16 due in MyLab tonight
Exam 1 Tues, Oct 5 8-9pm
in UC = Univ. Church

Monday: Review. See practice probs (1-12) at our home page at

www.math.purdue.edu/~bell/MA266

and all Exam 1 info (Exam cover page, map, seating chart, ...)





$$F_{Net} = ma$$

$$\underbrace{-ky}_{\text{Hooke's}} - \underbrace{cv}_{\text{Friction}} + \underbrace{F(t)}_{\text{external force}} = ma$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)$$ ← Non-homogeneous if $F \not\equiv 0$.

Case: No friction, no ext force:

$$m\ddot{y} + ky = 0$$ ← t missing. $p = \frac{dy}{dt}$

$$\frac{d^2y}{dt^2} = \frac{dp}{dt} = \frac{dp}{dy}\underbrace{\frac{dy}{dt}}_{p} = p\frac{dp}{dy}$$

$$mp\frac{dp}{dy} + ky = 0.$$ Separable

Better: $mr^2 + k = 0$

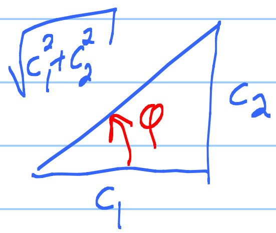
$$r^2 = -\frac{k}{m}$$

$$r = \pm\sqrt{\frac{k}{m}}\, i$$

Gen^l sol^n: $y = c_1 \cos \underbrace{\sqrt{\frac{k}{m}}}_{\omega_0} t + c_2 \sin \underbrace{\sqrt{\frac{k}{m}}}_{\omega_0} t$

Classic important trick:

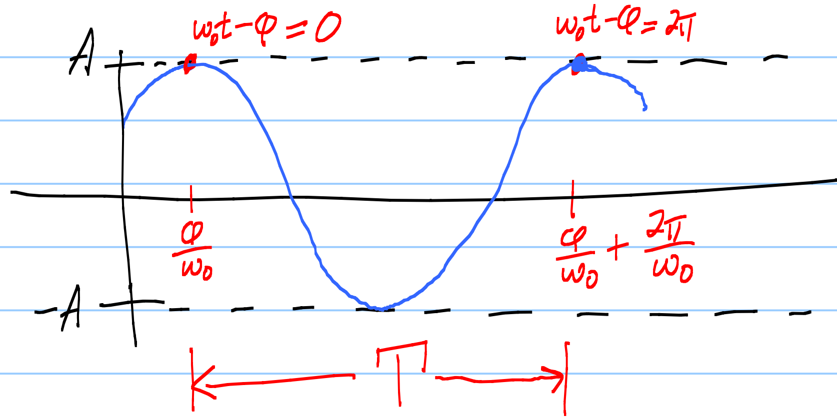
$$y = \underbrace{\sqrt{c_1^2+c_2^2}}_{A} \left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\varphi} \cos \omega_0 t + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\varphi} \sin \omega_0 t \right)$$



$$\boxed{= A \cos(\omega_0 t - \varphi)} \leftarrow \text{Easy to graph}$$

$\varphi = \operatorname{Tan}^{-1} \frac{c_2}{c_1} \leftarrow$ "phase shift"

$A = \sqrt{c_1^2+c_2^2} \leftarrow$ "amplitude"



T = "period" $= \frac{2\pi}{\omega_0}$ Frequency $= f = \frac{1}{T} = \frac{\omega_0}{2\pi}$

Trig identities: $e^{(\alpha+\beta)i} = e^{\alpha i} e^{\beta i} = (\cos\alpha + i\sin\alpha)(\cos\beta + i \sin\beta)$

$$\cos(\alpha \pm \beta) + i \sin(\alpha \pm \beta) = [\cos\alpha\cos\beta \mp \sin\alpha\sin\beta] + i[\cos\alpha\sin\beta \pm \cos\beta\sin\alpha]$$

Simple harmonic motion: (Undamped)

Damping: $m\ddot{y} + c\dot{y} + ky = 0$

$$mr^2 + cr + k = 0$$

Roots: $$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

<u>Case r_1, r_2 real and $\neq$:</u> $c^2 - 4mk > 0$

$$r_1 = \frac{-c}{2m} - \frac{\sqrt{c^2-4mk}}{2m} < 0$$ ← no brainer

<u>Claim:</u> $r_2 < 0$ too. Hmmm.

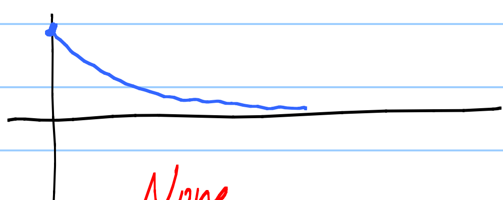
$$c^2 > c^2 - 4mk > 0 \qquad \sqrt{c^2} = c$$

So $$c > \sqrt{c^2 - 4mk} > 0$$

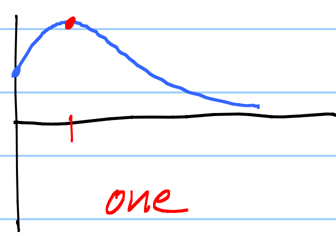
$$r_2 = \frac{\overbrace{-c + \sqrt{c^2-4mk}}^{<0}}{2m} < 0 \quad \text{too.}$$

Consequently, sol^n's $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ fizzle out fast!

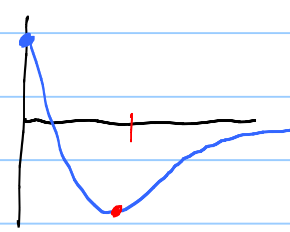
How many humps? None, or at most one:



None



one



one

<u>Overdamped</u> case. (Good shock absorbers.)

<u>Case $r_1 = r_2$:</u> $c^2 - 4mk = 0$

$$r_1 = r_2 = \frac{-c}{2m}$$

$$y = c_1 \underbrace{e^{-(\frac{c}{2m})t}} + c_2 \underbrace{t\, e^{-(\frac{c}{2m})t}}$$

tends to zero rapidly as $t \to \infty$

Critically damped case. (Shock absorbers about to go.)

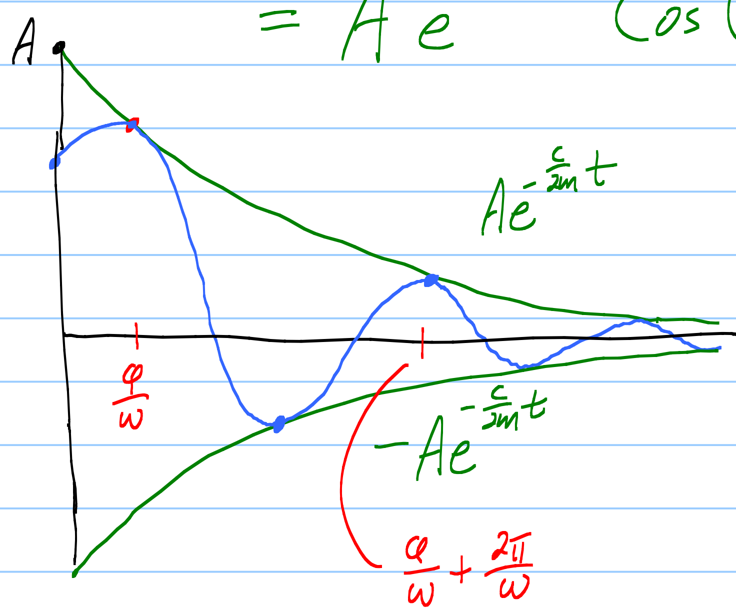
Same condition: No humps, or at most one.

Case r_1, r_2 complex $= \frac{-c}{2m} \pm \underbrace{\frac{\sqrt{|c^2-4mk|}}{2m}}_{\omega}\, i$

$(c^2 - 4km < 0)$

$$y = c_1 e^{\frac{-c}{2m}t} \cos \omega t + c_2 e^{-\frac{c}{2m}t} \sin \omega t$$

$$= A e^{-\frac{c}{2m}t} \cos(\omega t - \varphi)$$

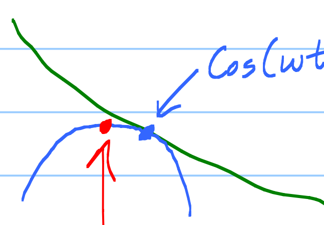


Shock absorbers shot!

$\frac{2\pi}{\omega}$ is a "pseudo period"

Underdamped case.

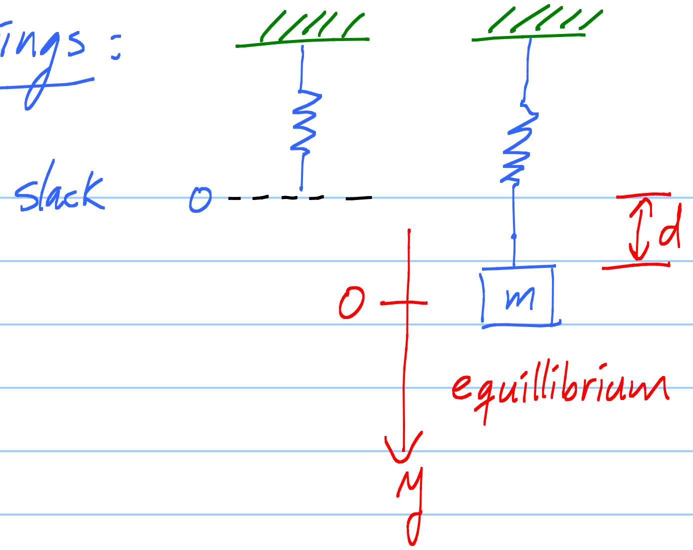
Top of humps?



$\cos(\omega t - \varphi)$ maxed out here

max! slightly before top of cos fcn

Vertical springs:



Hooke's: $kd = mg$

$$F = ma$$

$$-cv + mg - \underbrace{k(d+y)}_{-kd\,-\,ky} = m\frac{d^2y}{dt^2}$$

Cancel!

Get same ODE as sideways!