Review for Exam 1

Exam 1, Tues, Oct 5, 8:00-9:00pm
in Univ Church
over Lectures 1–16 up to 3.3(part 2)

1. If $\square\, y' + \underbrace{\left(1+\frac{1}{t}\right)}_{P(t)} y = \underbrace{\frac{1}{t}}_{Q(t)}$ and $y(1) = 0$, then $y(\ln 2) =$ Linear

In Standard Form! (Otherwise divide by $\square$ first.)

1. Linear
2. Separable
3. Homogeneous
4. Exact
5. Clever changes of vars. (Bernoulli equs)

Int. factor: $u = e^{\int P(t)dt} = e^{\int (1+\frac{1}{t})dt}$

$= e^{t+\ln|t|} = e^t e^{\ln|t|}$

$= e^t|t| = \pm t e^t$

Only need one int. fact. Take $\boxed{u = te^t}$

Mult Standard Form eqn by u:

$$\underbrace{(te^t)\left[y' + (1+\tfrac{1}{t})y\right]}_{[(te^t)y]'} = (te^t)\cdot\frac{1}{t} = e^t$$

$[\underbrace{(te^t)}_{u}\, y]'$

$\int e^{kt}\,dt$

$\int \frac{du}{u}$

$\int x^{\pm n}\,dx$

Sin, Cos

So $(te^t)y = \int e^t\,dt$

$= e^t + C$

$y = \frac{1}{t} + C\frac{e^{-t}}{t}$

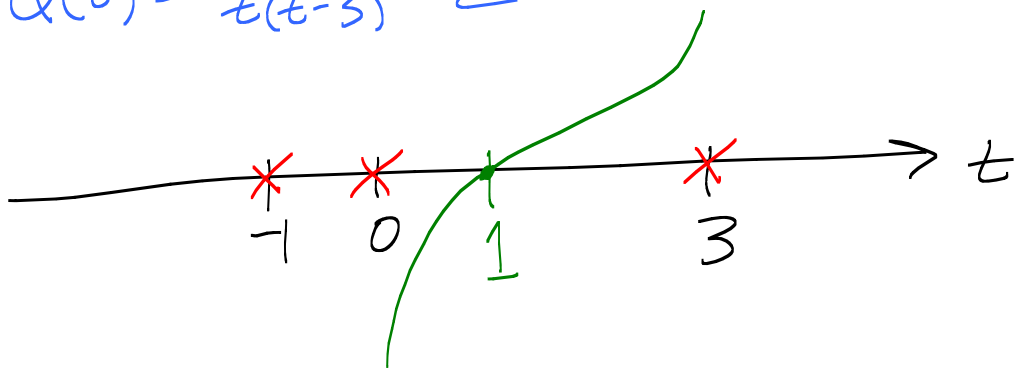
Plug in $t=1$, $y=0$: Get C. Finally $y(\ln 2)$.

2. What is the largest open interval for which a unique solution of the initial value problem $ty' + \frac{1}{t+1}y = \frac{t-2}{t-3}$, $y(1) = 0$ is guaranteed ?

zeroes dangerous!

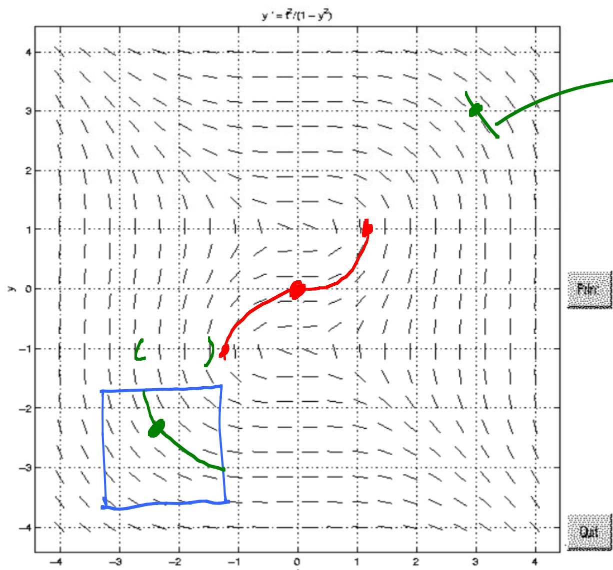
$$P(t) = \frac{1}{t(t+1)} \leftarrow \text{Boom! at } t = 0, -1$$

$$Q(t) = \frac{t-2}{t(t-3)} \leftarrow \text{Bad at } t = 0, 3$$



Bad!

3. Use the dfield plot below to estimate where the solution of $y' = \underbrace{\frac{t^2}{1-y^2}}_{f(t,y)}$, $y(0) = 0$ is defined:



line, slop $= f(3,3)$

Boom! at vertical tangent

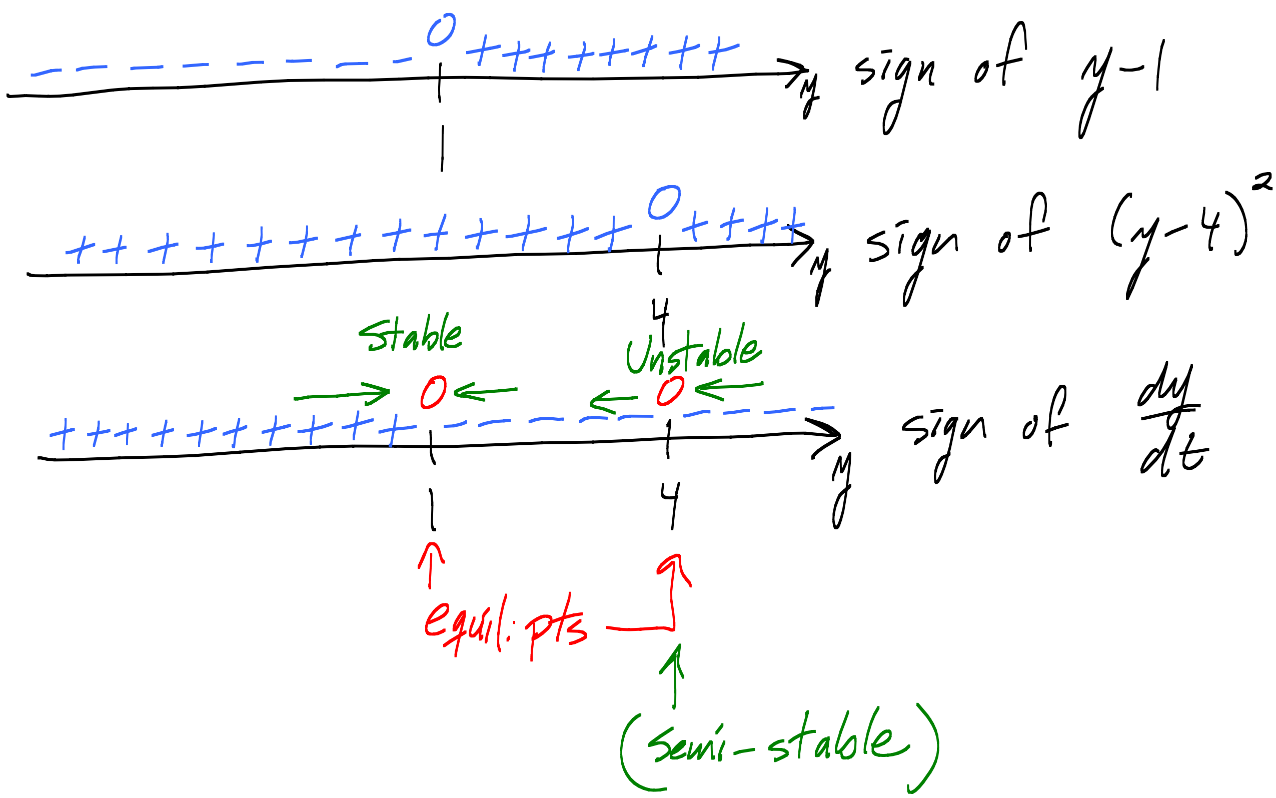
Non-linear eqns are strange: $\frac{dy}{dt} = y^2$

nice!

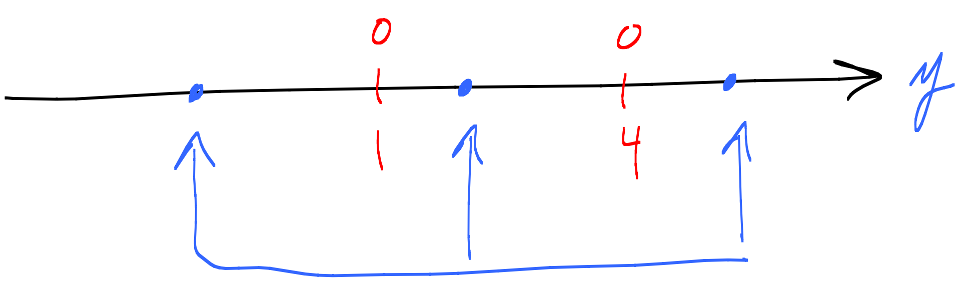
But sol^{n}s blow up in finite time.

no t on RHS

4. Consider the autonomous differential equation $\frac{dy}{dt} = -\frac{1}{10}(y-1)(y-4)^2$. Classify the stability of each equilibrium solution.



Or. Find equil. pts.



Test sign of RHS at one pt. in each interval.

(Intermediate value thm!)

Ouch! Not linear

5. Determine whether $\underbrace{x + 2y}_{M} + (\underbrace{2x + y}_{N})\,\frac{dy}{dx} = 0$ is separable, homogeneous, linear and/or exact.

$$M\,dx + N\,dy = 0$$

Exact: Is $\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$? $\frac{\partial}{\partial y}\left[\text{Thing by } dx\right] = \frac{\partial}{\partial x}\left[\text{Thing by } dy\right]$

$2 \overset{?}{=} 2$ Yes!

Not separable: $2y + y\frac{dy}{dx} = -x - 2x$. Stuck.

Linear: $\left[eee\right]\frac{dy}{dx} + \left[eee\right]y = \left[eee\right]$

Homog: $(\overset{10}{\overbrace{x^3y^7}} + \overset{10}{\overbrace{x^5y^5}}) + (\overset{10}{\overbrace{x^9y^1}} + \overset{10}{\overbrace{x^1y^9}})\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x+2y}{2x+y}\;\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \qquad \frac{\left(\frac{1}{x^{10}}\right)}{\left(\frac{1}{x^{10}}\right)}$$

$$= \frac{-1+2\left(\frac{y}{x}\right)}{2+\left(\frac{y}{x}\right)} \qquad \text{Let } v = \frac{y}{x}.$$

6. An explicit solution of $y' = y^2 - 1$ is

$$\int \frac{dy}{y^2-1} = \int dx \qquad \text{Sep.}$$

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

7. If $y' = y^3$ and $y(0) = 1$, then $y(-1) =$

Sep.

8. If $(2x^2+y^2)dx - xy\,dy = 0$ and $y(1)=2$, then $y(e^3) =$

Homog

$$\frac{dy}{dx} = \frac{2x^2+y^2}{xy} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$V = \frac{y}{x}$

So $\boxed{y = xV}$

can plug $y = xV$ and cancel x's

$$= \frac{2+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$

$$\boxed{\frac{dy}{dx} = 1 \cdot V + x\frac{dv}{dx}}$$

$$V + x\frac{dv}{dx} = \frac{2+V^2}{V} \leftarrow \text{Sep.}$$

9. An implicit solution of $\underbrace{y^2 + 1}_{M} + \underbrace{(2xy + 1)}_{N}\dfrac{dy}{dx} = 0$ is

$$\frac{\partial M}{\partial y} = 2y \overset{\text{Yes!}}{=} 2y = \frac{\partial N}{\partial x} \quad \text{Exact.}$$

Want φ with $\begin{cases} \dfrac{\partial \varphi}{\partial x} = M & (A) \\ \dfrac{\partial \varphi}{\partial y} = N & (B) \end{cases}$

(A): $\varphi = \int M\, dx = \int y^2 + 1\, dx = xy^2 + x + \underbrace{C(y)}_{\text{arb fcn of } y!}$

(B): $\dfrac{\partial \varphi}{\partial y} = \dfrac{\partial}{\partial y}\left(xy^2 + x + C(y)\right) \overset{\text{want}}{=} \underbrace{2xy + 1}_{N}$

$$\cancel{2xy} + 0 + C'(y) = \cancel{2xy} + 1$$

$$C'(y) = 1$$

$$C(y) = \int 1\, dy = y$$

Sol^n: $\varphi(x, y) = K$

$$\boxed{xy^2 + x + y = K}$$

← plug in Initial values to get K

10. If y' is proportional to y, $y(0) = 2$ and $y(1) = 8$. For what value of t does $y(t) = 20$?

11. The general solution of $y'' - 4y' + 4y = 0$ is

12. The general solution of $y''' + 4y'' + 5y' = 0$ is

18. A tank initially contains 40 ounces of salt mixed in 100 gallons of water. A solution containing 4 oz of salt per gallon is then pumped into the tank at a rate of 5 gal/min. The stirred mixture flows out of the tank at the same rate. How much salt is in the tank after 20 minutes ?