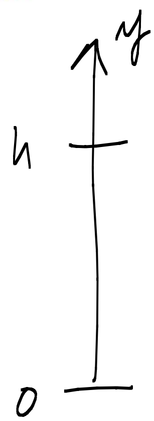
Exam 1 solutions

HW17 due Wed in MyLab
HW16w,17w due in GS

No class Monday, Oct Break

1. *(10 pts.)* A stone is dropped from rest at an initial height $h = 25$ feet above the ground. Ignoring air resistance, assume that the acceleration due to gravity is $g = 32$ ft/sec^2. How long does it take for the stone to hit the ground?

1.25 sec



$$m\frac{d^2y}{dt^2} = -mg$$

$$\frac{dy}{dt} = -\int g\,dt = -gt + \overset{\overset{0}{\|}}{V_0}$$

$$y = -\int gt\,dt = -\frac{1}{2}gt^2 + h$$

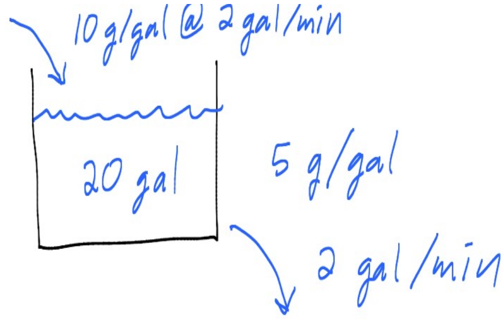
When is $y = -\frac{1}{2}gt^2 + h = 0$?

$$25 = 16t^2$$

$$t = \sqrt{\frac{25}{16}} = \frac{5}{4} \text{ sec}$$

2. A fish tank contains 20 gallons of a salt solution with a concentration of 5 grams of salt per gallon. A salt solution with a concentration of 10 grams/gallon is added to the tank at a rate of 2 gallons per minute. At the same time, well-mixed water is drained from the tank at a rate of 2 gallons per minute. How many grams of salt are in the tank after 10 minutes?

$$200 - 100e^{-1}$$



$S(t) =$ g of salt in tank @ time t

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= 10 \cdot 2 - 2 \cdot \frac{S}{20}$$

$$\frac{dS}{dt} + \underbrace{\frac{1}{10}}_{P(t)} S = 20$$

$$\text{Int. Fact} \quad e^{\int P(t)\,dt} = e^{\int \frac{1}{10}\,dt} = e^{t/10}$$

$$\underbrace{e^{t/10}\left(S' + \tfrac{1}{10}S\right)}_{\left[e^{t/10} S\right]'} = 20e^{t/10}$$

$$e^{t/10} S = \int 20\, e^{t/10}\, dt = 20 \frac{1}{(1/10)} e^{\frac{1}{10}t} + C$$

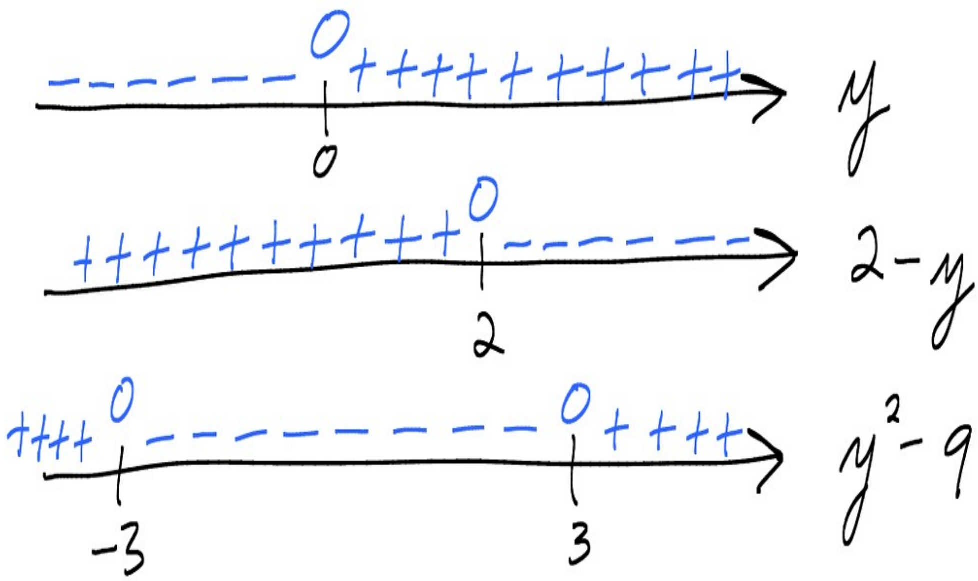
Plug $t=0$ $\quad 1 \cdot 100 = 200 \cdot 1 + C \quad \boxed{C = -100}$

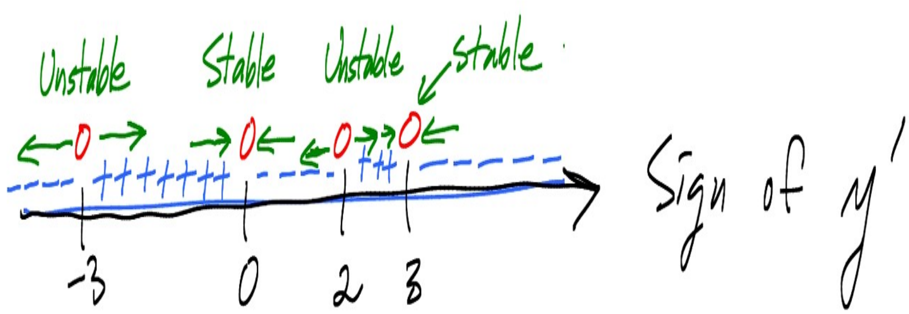
$S(t) = 200 - 100e^{-t/10} \qquad S(10) = 200 - 100e^{-1}$

3. *(10 pts.)* Which one of the following statements is true about equilibrium solutions to

$$y' = y(2-y)\left(y^2-9\right)?$$

$y = 0$, $y = 3$ are stable; $y = 2$ and $y = -3$ are unstable.





4. *(10 pts.)* Find the general solution to the differential equation

$$y'' + 4y' + 4y = 0$$

$$y = c_1e^{-2t} + c_2te^{-2t}$$

Try e^{rt} :

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$r = -2, -2$ repeated!

$$y = c_1e^{-2t} + c_2te^{-t}$$

5. *(10 pts.)* For the initial value problem $y' = t^2 + y^2$, $y(1) = 2$, use the Euler method with $h = 0.5$ to find an approximate value of $y(2)$.

15.75

$$\begin{cases} y_{n+1} = y_n + h f(x_n, y_n) \\ x_{n+1} = x_n + h \end{cases}$$

$$\begin{cases} x_0 = 1 \\ y_0 = 2 \end{cases}$$

$x_1 = x_0 + \frac{1}{2} = \frac{3}{2}$

$y_1 = y_0 + \frac{1}{2} f(x_0, y_0)$

$= 2 + \frac{1}{2}(\underbrace{x_0^2 + y_0^2}_{1^2 + 2^2}) = 2 + \frac{5}{2} = \frac{9}{2}$

$x_2 = 2$

$y_2 = y_1 + \frac{1}{2} f(x_1, y_1) = \frac{9}{2} + \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + \left(\frac{9}{2}\right)^2\right)$

$= \frac{9}{2} + \frac{90}{8} = \frac{126}{8} = 15\frac{3}{4}$

6. *(10 pts.)* Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{xy^2}{x^2+1}, \qquad y(0) = 3.$$

$$y = \frac{6}{2 - 3\ln(1+x^2)}$$

$$\int \frac{dy}{y^2} = \int \frac{x}{\underbrace{x^2+1}_{u}}\, dx \qquad du = 2x\,dx$$

$$\frac{1}{-2+1} y^{-2+1} = \int \frac{\frac{1}{2}du}{u} = \ln|u| + C$$

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) + C$$

Plug $x=0$, $y=3$.

$$-\frac{1}{3} = 0 + C$$

$$\boxed{C = -\frac{1}{3}}$$

$$y = \frac{-1}{\frac{1}{2}\ln(x^2+1) - \frac{1}{3}} \cdot \frac{6}{6}$$

7. *(10 pts.)* The change of variables $v = y/x^2$ transforms the equation

$$\frac{dy}{dx} = \sin(y/x^2) \qquad \text{into}$$

$$2xv + x^2v' = \sin(v)$$

$$V = \frac{y}{x^2}$$

$$y = x^2 \cdot V$$

$$\frac{dy}{dx} = \underbrace{2x \cdot V + x^2 \frac{dV}{dx} = \sin(V)}_{\text{New ODE}}$$

8. *(10 pts.)* Solve the initial value problem

$$2\frac{dy}{dx} + 3y = e^{2x}, \qquad y(0) = 1.$$

$$\boxed{\frac{dy}{dx} + \underbrace{\frac{3}{2}}_{P(t)} y = \frac{1}{2} e^{2x}}$$ Stand. form!

Int. Fact: $u = e^{\int \frac{3}{2} dx} = e^{\int \frac{3}{2} dx} = e^{\frac{3}{2}x}$

$$\underbrace{e^{\frac{3}{2}x}\left(y' + \frac{3}{2}y\right)}_{\left[e^{\frac{3}{2}x} y\right]'} = \frac{1}{2}e^{2x} \cdot e^{\frac{3}{2}x}$$

$$e^{\frac{3}{2}x} y = \int \frac{1}{2} e^{\frac{7}{2}x} \, dx$$

$$e^{\frac{3}{2}x} y = \frac{1}{2} \frac{1}{\left(\frac{7}{2}\right)} e^{\frac{7}{2}x} + C$$

$y(1) = 2$
$\{x = 1, y = 2\}$

$$\boxed{y = \underbrace{\frac{1}{7}e^{2x}}_{} + \underbrace{Ce^{-\frac{3}{2}x}}_{}}$$

particular sol^n
$y' + \frac{3}{2}y = e^{2x}$

Gen^l Sol^n to homog equ
$y' + \frac{3}{2}y = 0$

Get $C = \frac{6}{7}$

$y = \frac{1}{7}e^{2x} + \frac{6}{7}e^{-\frac{3}{2}x}$

9. *(10 pts.)* Explain why the following equation is exact and find an implicit formula for the solution to the initial value problem

$$\underbrace{\left(\frac{y}{x}+6x\right)}_{M}+\underbrace{(\ln x-2y)}_{N}\frac{dy}{dx}=0, \qquad y(1)=2; \quad x>0. \qquad M\,dx+N\,dy=0$$

Exact? $\quad \dfrac{\partial M}{\partial y} \overset{?}{=} \dfrac{\partial N}{\partial x}$

$\dfrac{1}{x} \overset{?}{=} \dfrac{1}{x}$ ✓ Yes!

Find φ with $\begin{cases} \dfrac{\partial \varphi}{\partial x} = M & (A) \\ \dfrac{\partial \varphi}{\partial y} = N & (B) \end{cases}$ $\qquad$ Sol^n: $\varphi(x,y)=C$

(A): $\varphi = \int \frac{y}{x} + 6x\,dx = y\ln x + 3x^2 + \underbrace{C(y)}_{\text{arb fcn } y}$

(B): $\dfrac{\partial}{\partial y}\Big[\underbrace{y\ln x + 3x^2 + C(y)}_{\varphi}\Big] \overset{\text{want}}{=} \underbrace{\ln x - 2y}_{N}$

$$\ln x + 0 + C'(y) = \ln x - 2y$$

$$C'(y) = -2y$$

So $C(y) = -y^2$

Got φ!

$$\varphi(x,y) = y\ln x + 3x^2 - y^2 = K \quad \leftarrow \text{Gen}^l \text{ Sol}^n$$

$y(1)=2 \qquad x=1, y=2: \qquad 2\cdot 0 + 3\cdot 1^2 - 2^2 = K$

$3-4=K \qquad \boxed{K=-1}$

$y\ln x + 3x^2 - y^2 = -1$

Bad reasoning:

$$\varphi = y\ln x + 3x^2 + f(y) = y\ln x - y^2 + g(x)$$

Hmmm. $f(y) + y^2 = g(x) - 3x^2 = C$

Plug $x=1$. See LHS = const.

Plug $y=2$. See RHS = same const.

$$f(y) = C - y^2$$
$$g(x) = C + 3x^2$$

Works!

10. *(10 pts.)* Solve the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = 1.$$

$r^2 - r - 2 = 0 \leftarrow$ 2pts.

$(r-2)(r+1) = 0 \quad r = -1, 2 \leftarrow$ 2pts.

$y = c_1 e^{-t} + c_2 e^{2t} \leftarrow$ 2pts

$y' = -c_1 e^{-t} + 2c_2 e^{2t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= -c_1 + 2c_2 = 1 \end{aligned}\right\} \leftarrow \text{2pts}$$

$3c_2 = 2 \qquad c_2 = \frac{2}{3} \leftarrow$ 1pt.

$c_1 = 1 - c_2 = \frac{1}{3} \leftarrow$ 1 pt.

$y = \frac{1}{3} e^{-t} + \frac{2}{3} e^{2t}$