Lecture 18 on 3.5 The method of undetermined coefficients

HW17 due MyLab
HW16W, 17W due in GS

Big fact: $(\dagger)\quad y'' + P(x)y' + Q(x)y = F(x) \leftarrow$ non-homog

$(*)\quad y'' + P(x)y' + Q(x)y = 0 \leftarrow$ homog

Gen^l Sol^n to $(\dagger)$ is

$$\underbrace{c_1 y_1 + c_2 y_2}_{\substack{\text{gen}^l \text{ sol}^n \text{ to } (*) \\ (\text{homog sol}^n) \\ y_c \text{ the "complementary sol}^n\text{"}}} + y_p$$

$y_p \leftarrow$ One particular sol^n to $(\dagger)$

Why: $L[y] = y'' + P(x)y' + Q(x)y$ is a linear operator

Suppose y_p is a part sol^n to $(\dagger)$ and Y is any other sol^n to $(\dagger)$. Then

$$L[y_p] = F$$
$$-\quad L[Y] = F$$

$$L[\underbrace{Y - y_p}_{\text{solves } (*)!}] = F - F = 0$$

So $Y - y_p = c_1 y_1 + c_2 y_2$ for some c's.

$$Y = c_1 y_1 + c_2 y_2 + y_p \quad \checkmark$$

Last thing to check. These all solve $(\dagger)$:

$$L[c_1 y_1 + c_2 y_2 + y_p] = \underbrace{L[c_1 y_1 + c_2 y_2]}_{=0} + \underbrace{L[y_p]}_{F}$$

$$= F \checkmark$$

EX: $y'' + 4y = e^{-2x}$

Step 1: Solve homog equ: $y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

Step 2: Find a y_p. Guess $y_p = Ae^{-2x}$

Try it:

$y_p' = -2Ae^{-2x}$

$y_p'' = 4Ae^{-2x}$

Plug into (+) and force it.

Want $y_p'' + 4y_p = e^{-2x}$

$$(4Ae^{-2x}) + 4(Ae^{-2x}) \overset{\text{want}}{=} e^{-2x}$$

$$\underbrace{8A}_{} e^{-2x} = e^{-2x}$$

Aha! Need $8A = 1$ $\boxed{A = 1/8}$

Get $y_p = \frac{1}{8}e^{-2x}$ and gen^l sol^n to (+) is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}e^{-2x}$$

Method of Undetermined Coefficients

p. 191

$$\underbrace{ay'' + by' + cy}_{\text{const coeff linear}} = \underbrace{F(x)}_{\text{special form}}$$

$F(x)$	Try $y_p =$
e^{rx}	Ae^{rx}
x^n or any poly of deg n	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$
$\cos wx$ $\sin wx$ $a\cos wx + b\sin wx$	$A\cos wx + B\sin wx$
$e^{rx}\cos wx$ $e^{rx}\sin wx$	$Ae^{rx}\cos wx + Be^{rx}\sin wx$
$x^n e^{rx}\cos wx$ $x^n e^{rx}\sin wx$ $P_n(x)e^{rx}\cos wx$ poly deg n	$(A_n x^n + \cdots + A_1 x + A_0)e^{rx}\cos wx +$ $(B_n x^n + \cdots + B_1 x + B_0)e^{rx}\sin wx$ $B_3 x^3 e^{rx}\sin wx$ is a "single piece"

Rule: But, if any single piece of "trial sol^n" from the table solves the homog prob, try x^s (trial sol^n y_p from table)

where s is the smallest positive integer so that no single term in the new trial sol^n solves the homog.

EX: $y'' + y = \cos x$ ← Resonance!

Step 1: Solve $y'' + y = 0$

$r^2 + 1 = 0, \quad r = \pm i$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

Step 2: From table: trial sol^n $\underbrace{A\cos x + B\sin x}_{\text{ouch! Solves } (*)}$.

Method bombs!

Rule: Try $y_p = x(A\cos x + B\sin x)$

Single pieces: $x\cos x$, $x\sin x$ do not solve $(*)$

$s=1$ is smallest power that clears the danger.

$$y_p' = 1\cdot(A\cos x + B\sin x) + x(-A\sin x + B\cos x)$$

$$y_p'' = 0\cdot(A\cos x + B\sin x) + 2\cdot 1\cdot(-A\sin x + B\cos x) + x\cdot(-A\cos x - B\sin x)$$

$(uv)'' = u''v + 2u'v' + uv''$

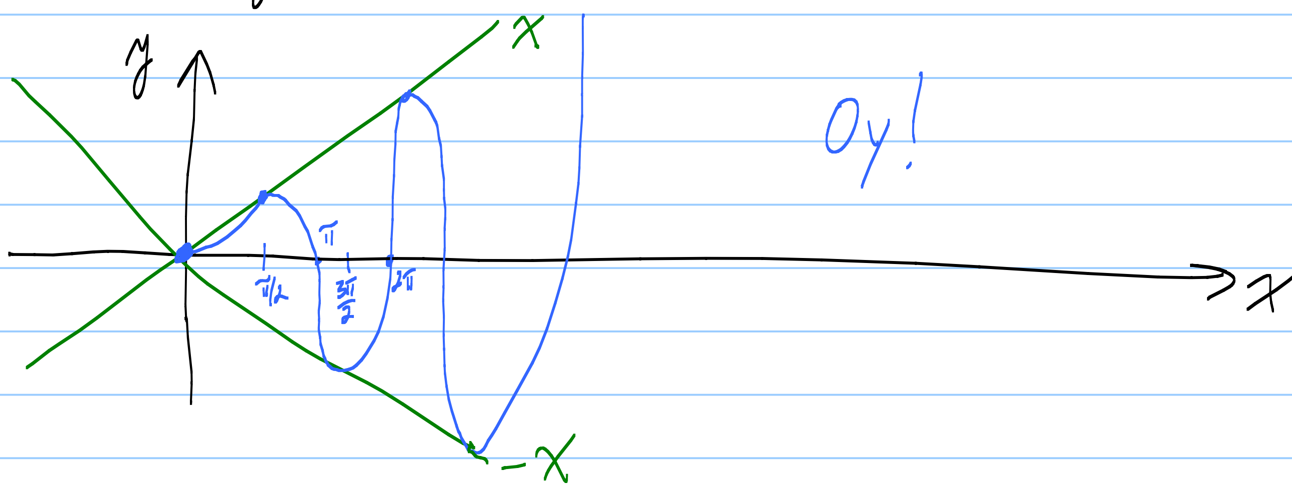
Plug into (†):

$$\underbrace{\left[(-2A\sin x + 2B\cos x) - x(A\cos x + B\sin x)\right]}_{y_p''} + \underbrace{\left[x(A\cos x + B\sin x)\right]}_{y_p} \overset{\text{want}}{=} \cos x$$

$$\text{need}\quad \underbrace{-2A}_{=0}\sin x + \underbrace{2B}_{=1}\cos x = \cos x$$

Get $A=0$, $B=\frac{1}{2}$ works!

So $y_p = x\left(0\cdot\cos x + \frac{1}{2}\sin x\right) = \frac{1}{2}x\sin x$



Oy!

Gen'l sol'n to (†): $\underbrace{c_1\cos x + c_2\sin x}_{A\cos(x-\varphi)} + \frac{1}{2}x\sin x$

Oscillations explode!

EX: $y'' + 4y' + 4y = e^{-2x}$

Homog: $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2, -2$ repeated

$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

From table: trial $\underbrace{Ae^{-2x}}_{\text{ouch!}}$

Next try: $\underbrace{x(Ae^{-2x})}_{\text{also ouch!}}$

Finally: $\boxed{x^2 A e^{-2x}}$ is the correct trial sol^n

Fun thing: $(D^2 + 4D + 4)y = e^{-2x}$

$(D+2)^2 y = e^{-2x}$ ← hit this eq^n with $D+2$, which "annihilates e^{-2x}"

$$(D+2)(D+2)^2 y = \underbrace{(D+2)e^{-2x}}_{=0}$$

$(D+2)^3 y = 0$

$(r+2)^3 = 0$

$r = -2, -2, -2$ repeated 3 times.

So $$y = \underbrace{c_1 e^{-2x} + c_2 x e^{-2x}}_{\text{Aha! Gen}^l\text{ sol}^n\text{ to homog}} + \underbrace{c_3 x^2 e^{-2x}}_{\text{Must be a particular sol}^n\text{!}}$$

HWK prob: $$y^{(5)} + 5y^{(4)} - y = 17$$

$$r^5 + 5r^4 - 1 = 0 \leftarrow \text{Oy!}$$

Can't find homog sol^n. Table: $17 =$ poly of deg 0

Trial sol^n: $y_p = A_0$

does not solve homog