Lecture 19 3.5 (part 2) Variation of Parameters

HW18 due in MyLab 1

EX: $(D^2+1)^2 y = x\cos x$ ← (poly deg 1)·(Cos or Sin)

$y^{(4)} + 2y'' + y = x\cos x$

$(r^2+1)^2 = 0 \quad r = \pm i$ repeated!

$e^{ix}, \; xe^{ix}$

$\cos x, \; \sin x, \; x\cos x, \; x\sin x$

$y_c = c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x$

Table: Trial $y_p =$
$(A_1x+A_0)\cos x + (B_1x+B_0)\sin x$

piece: $B_1 x\sin x$
solves homog!

Mult. by x^s:

x^1: piece: $A_0 x\cos x$
ouch!

Correct guess: $x^2\left[(A_1x+A_0)\cos x + (B_1x+B_0)\sin x\right]$

Variation of Parameters

$$\underbrace{y'' + P(x)y' + Q(x)y}_{\text{not const. coeff}} = F(x)$$

not const. coeff **or** not of special form

Big assumption: We know gen'l sol'n to the homog eqn:

$$y_c = c_1 y_1 + c_2 y_2$$

parameters

Get a particular sol'n $\quad y_p = u_1 y_1 + u_2 y_2$

fcns

where $\quad u_1' = \dfrac{-y_2 F}{W[y_1, y_2]}, \quad u_2' = \dfrac{y_1 F}{W[y_1, y_2]}$

Danger! Using formula requires the F from the standard form eqn!

Why: Try $y_p = u_1 y_1 + u_2 y_2$, plug in ODE, force.

(Hmmm. When we do this, only get one equ for two u's. We need 2 equs.)

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{\text{take } = 0\,!} + u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug In:

$$\left[u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''\right] + P\left[u_1 y_1' + u_2 y_2'\right] + Q\left[u_1 y_1 + u_2 y_2\right] \overset{\text{want}}{=} F$$

red stuff: $u_1 L[y_1] = 0$

green stuff: $u_2 L[y_2] = 0$

$$\begin{array}{ll} \text{First equ:} & \left\{ \begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2 = F \end{array} \right. \\ \text{Second equ:} & \end{array}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Cramer's rule:

$$u_1' = \frac{\det \begin{bmatrix} 0 & y_2 \\ F & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \text{formula} \checkmark$$

$$u_2' = \frac{\det\begin{bmatrix} y_1 & 0 \\ y_1' & F \end{bmatrix}}{\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \text{formula} \checkmark$$

EX: $y'' + y = \underset{\uparrow \text{ not "special"}}{\operatorname{Sec} x}$ $\qquad F(x) = \operatorname{Sec} x$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

$$W = \det\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos^2 x - (-\sin^2 x) \equiv 1$$

$$u_1' = \frac{-y_2 F}{W} = \frac{-\sin x \overset{1/\cos x}{\operatorname{Sec} x}}{1} = -\operatorname{Tan} x$$

$$\text{So} \quad u_1 = \int -\operatorname{Tan} x \, dx = -\operatorname{Ln}|\operatorname{Sec} x| \underbrace{+ C}_{\substack{\text{dump } +C \\ \text{Just want } u\text{'s that} \\ \text{work.}}}$$

$$= \operatorname{Ln}|\operatorname{Sec} x|^{-1} = \underline{\underline{\operatorname{Ln}|\cos x|}}$$

$$u_2' = \frac{y_1 F}{W} = \frac{\cos x \operatorname{Sec} x}{1} = 1$$

$$\text{So} \quad u_2 = \int 1 \, dx = x \qquad (\text{no } +C)$$

Get $y_p = u_1 y_1 + u_2 y_2 = \underbrace{(\operatorname{Ln}|\cos x|)}_{u_1} \underbrace{\cos x}_{y_1} + \underbrace{(x)}_{u_2} \cdot \underbrace{\sin x}_{y_2}$

Gen^l sol^n: $c_1 \cos x + c_2 \sin x + y_p$

EX: $x^2 y'' - 4xy' + 4y = -2x^2$

Euler eqn.
HWK: $v = \ln x$ turns into a const coeff homog eqn:
$$a\frac{d^2y}{dv^2} + b\frac{dy}{dv} + cy = 0$$

Or reasonable to guess and try $y = x^r$ for homog sol^n.

$$y' = rx^{r-1}$$
$$y'' = r(r-1)x^{r-2}$$

Plug in $x^2\left[r(r-1)x^{r-2}\right] - 4x\left[rx^{r-1}\right] + 4\left[x^r\right] = 0$

$$\Big[\underbrace{r(r-1) - 4r + 4}_{\text{must } = 0}\Big]\underbrace{x^r}_{\text{not zero, } x \neq 0} = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-1)(r-4) = 0$$

Great! Get $y_1 = x^1 = x \qquad y_2 = x^4$

$\frac{y_2}{y_1}$ is not const. So y_1, y_2 are lin. ind.

$$W = \det\begin{bmatrix} x & x^4 \\ 1 & 4x^3 \end{bmatrix} = 3x^4$$

Hmmm. $W(0) = 0$, Abel's Thm?

Standard form: $y'' - \frac{4x}{x^2}y' + \frac{4}{x^2}y = \frac{-2x^2}{x^2} = \underbrace{-2}_{F(x)}$

ouch at $x = 0$!

Safe, on intervals that don't contain $x=0$.

$$u_1' = \frac{-x^4(-2)}{3x^4} = \frac{2}{3} \qquad \boxed{u_1 = \frac{2}{3}x}$$

$$u_2' = \frac{x(-2)}{3x^4} = -\frac{2}{3}x^{-3} \qquad u_2 = \frac{1}{(-3+1)}\left(-\frac{2}{3}\right)x^{-3+1}$$

$$\boxed{u_2 = \frac{1}{3}x^{-2}}$$

Get $y_p = u_1 y_1 + u_2 y_2$, etc.

Euler eqns: Cases r_1 repeated roots or complex roots.

Very interesting! $\quad x^{a+bi} = ?$

$$\frac{\partial}{\partial r}\left[x^r\right] = ?$$

Important fact: $\underbrace{A(x)y'' + B(x)y' + C(x)y}_{L[y]} = F_1(x) + F_2(x) + F_3(x)$

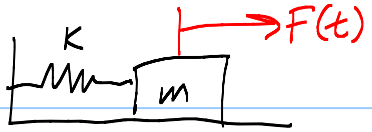
Get $y_{p_1}, y_{p_2}, y_{p_3}$ with $L[y_{p_k}] = F_k \quad k=1,2,3$

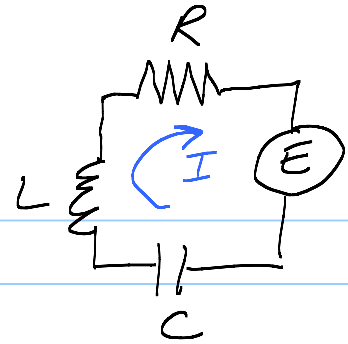
Then $y_p = y_{p_1} + y_{p_2} + y_{p_3}$

Why: L is linear

$$L[y_{p_1} + y_{p_2} + y_{p_3}] = \sum_{k=1}^{3} L[y_{p_k}] = \sum_{k=1}^{3} F_k$$

Forced vibrations:





$$my'' + cy' + ky = F$$

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{E}(t)$$

non-homogeneous equns!