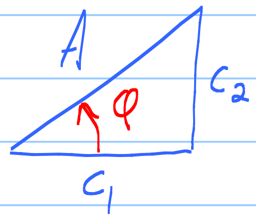
Lecture 20 3.6 Forced vibrations

HW 19 due in MyLab

$$y = c_1 \cos \omega t + c_2 \sin \omega t = A \cos(\omega t - \varphi)$$



$y(0) = c_1 = y_0$

$y'(0) = -c_2 \omega = y_0'$

$\varphi = \tan^{-1} \frac{c_2}{c_1}$, $A = \sqrt{c_1^2 + c_2^2}$

EX: $y = 3\cos t - 4 \sin t$

$\begin{cases} y(0) = 3 \\ y'(0) = 4 \end{cases}$ $A = \sqrt{3^2+4^2} = 5$, $\varphi = \tan^{-1}\left(\frac{-4}{3}\right)$

or

(1) $y(0) = A\cos(-\varphi) = A\cos\varphi = y_0$ (want)

$y'(t) = -A\omega \sin(\omega t - \varphi)$

(2) $y'(0) = -A\omega \sin(-\varphi) = A\omega\sin\varphi = y_0'$ (want)

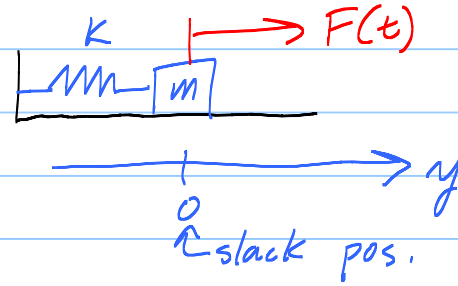
$\frac{(2)}{(1)} = \frac{A\omega \sin\varphi}{A\cos\varphi} = \frac{y_0'}{y_0}$

$\tan\varphi = \frac{y_0'}{\omega y_0}$

Got φ. Now get A from (1).

Forced vibrations

Undamped case: $m\ddot{y} + ky = F(t)$



$m\ddot{y} + ky = F_0 \cos \omega t$ ← "input"

Step 1: Solve homog equ:

$$m\ddot{y} + ky = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}\, i$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$.

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t = A \cos(\omega_0 t - \varphi)$$

Step 2: Find a part. sol^n y_P by Meth. of Undet. Coeff.

Case $\omega \neq \omega_0$: Try $y_P = A\cos\omega t + B\sin\omega t$

$$y_P' = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$y_P'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

Plug in ODE:

$$m\left[-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right] + k\left[A\cos\omega t + B\sin\omega t\right]$$

$$\underbrace{(-mA\omega^2 + kA)}_{=F_0}\cos\omega t + \underbrace{(-mB\omega^2 + kB)}_{=0}\sin\omega t \overset{\text{want}}{=} F_0\cos\omega t$$

so $A = \dfrac{F_0}{k - m\omega^2}$ So $B = 0$

$$A = \frac{F_0}{m\left(\frac{k}{m} - \omega^2\right)} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

ω_0^2

Get $y_P = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$

Whoa! if $\omega \approx \omega_0$, this gets gigantic!

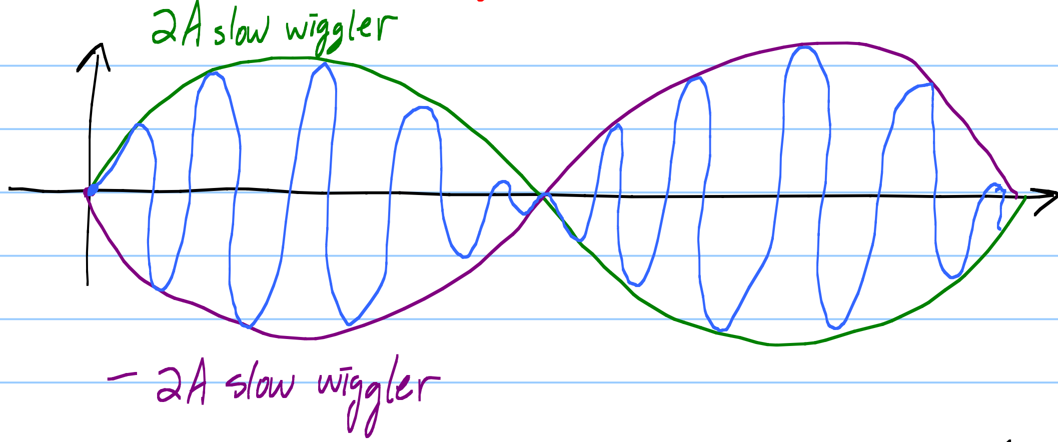
Gen^l sol^n: $A\cos(\omega_0 t - \varphi) + \dfrac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$

EX: $y = A(\cos\omega t - \cos\omega_0 t)$

$$= 2A\sin\frac{(\omega_0 - \omega)t}{2}\sin\frac{(\omega_0 + \omega)t}{2}$$

ω close to ω_0

slow wiggler　　　　fast wiggler close to ω_0

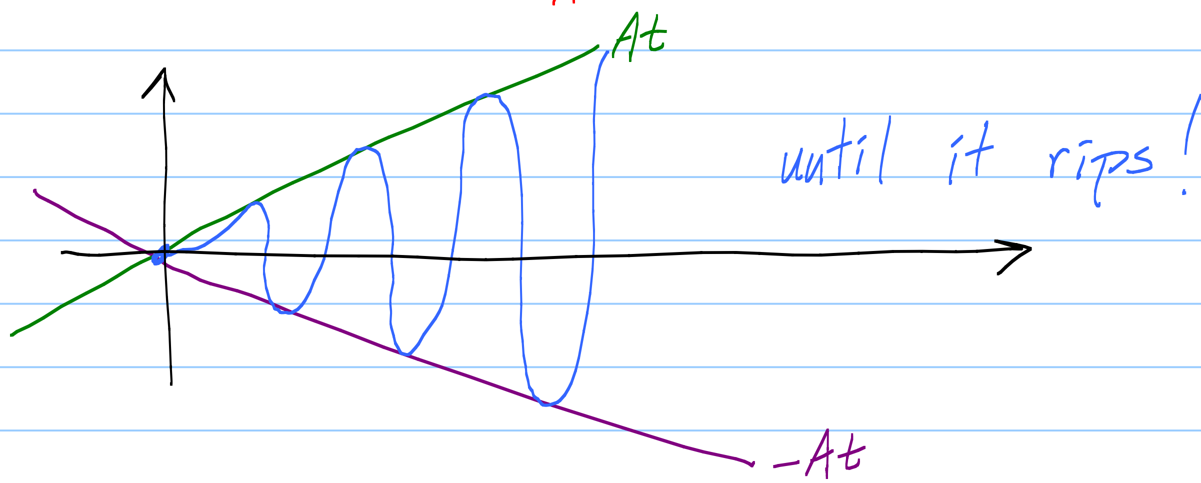


Case $\omega = \omega_0$　　Resonance!　　Try $y_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t)$

EX:　$my'' + ky = F_0 \cos\omega_0 t$

Get $y_p = \dfrac{F_0}{2m\omega_0}\, t \sin\omega_0 t$

$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$　　$y = \underbrace{\dfrac{F_0}{2m\omega_0}}_{A}\, t \sin\omega_0 t$



Damping　　$m\ddot{y} + c\dot{y} + ky = F_0 \cos\omega t$

Step 1: Homog prob:　$mr^2 + cr + k = 0$

$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

Underdamped: $c^2 - 4km < 0$

$$r = \frac{-c}{2m} \pm \underbrace{\frac{\sqrt{|c^2 - 4km|}}{2m}}_{\mu} i$$

Homog sol^n:

$$a e^{-\frac{c}{2m}t} \cos(\mu t - \varphi)$$

Step 2: $y_p = \underbrace{A \cos \omega t + B \sin \omega t}_{\text{safe!}}$

Plug in. Get linear eqns in A, B. Solve

Get $$\boxed{y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \eta)} \quad (*)$$

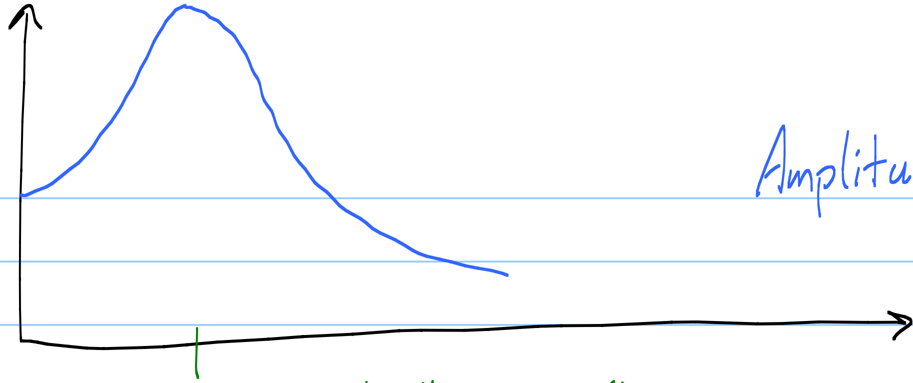
Overdamped: Homog sol^n $c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$ negative exponentials

Critically damped: Homog sol^n $c_1 e^{-rt} + c_2 t e^{-rt}$

Gen^l Sol^n: $y = \underbrace{(\text{homog sol}^n)}_{\text{transient terms}} + (*)$ ← particular

neg exponentials make them fizzle out fast

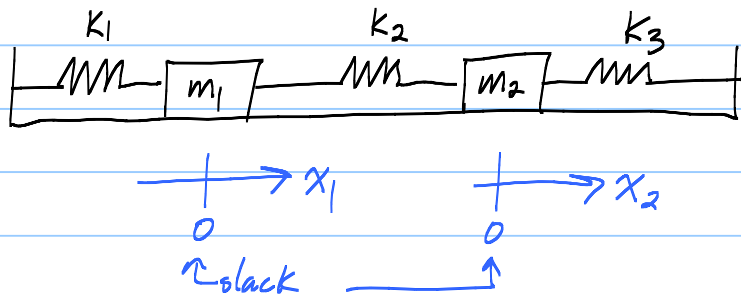
"response"



Amplitude of response

as close to "resonance" as you can get

Fun problem:



$F=ma$ for m_1: $\quad m_1 \ddot{x}_1 = \text{net force on } m_1$

$$= -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$