Lecture 21 3.6 (part 2), 4.1

HW 20 due in MyLab

HW 18w, 19w due in GS

$$my'' + cy' + ky = F_0 \cos \omega t \leftarrow \text{"input"}$$

$$y = \underbrace{\begin{cases} c_1 e^{-r_1 t} + c_2 e^{-r_2 t} & (O) \\ c_1 e^{-rt} + c_2 t e^{-rt} & (C) \\ A e^{\frac{-c}{2m}t} \cos(\mu t - \varphi) & (U) \end{cases}}_{\substack{\text{homog sol}^n \\ \text{Transient sol}^n}} + \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \eta)}_{\substack{y_p \text{ part. sol}^n \\ \text{"Response"}}}$$

Where $\omega_0 = \sqrt{\frac{k}{m}}$, $\mu = \frac{\sqrt{|c^2-4km|}}{2m}$

Notation: $\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}$, $\eta = \sin^{-1} \frac{c\omega}{\Delta}$

Prob: What ω gives biggest amplitude of response?

$\frac{F_0}{\sqrt{}}$ is max when $\sqrt{}$ is min, and

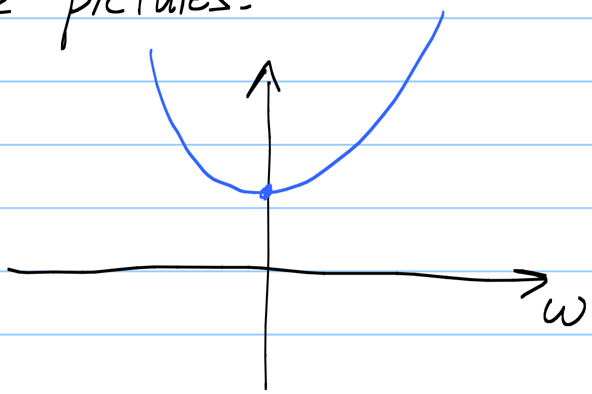
$\sqrt{}$ is min when $\underbrace{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}_{\text{4th deg poly in } \omega.}$ is min

Fresh Calc problem!

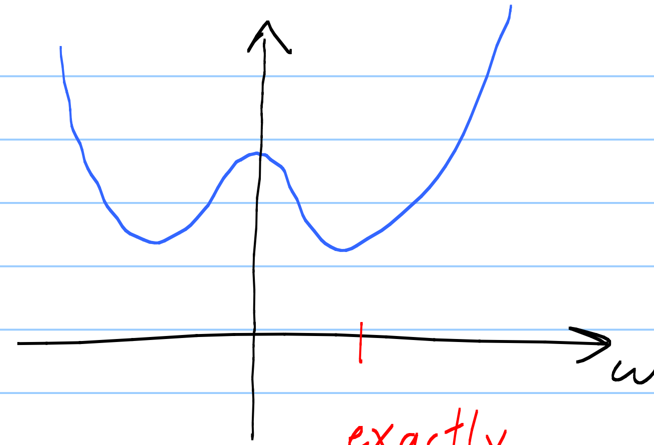
Properties:
1) > 0
2) $\to \infty$ as $\omega \to \pm\infty$
   so it has one or more mins, maybe up to ? (2,3)
3) It is symmetric about vertical axis.
4) Derivative is a 3rd degree poly

So deriv $= 0$ at 1 or 3 places $\leftarrow$ extrema.

Possible pictures:



no min for $\omega > 0$



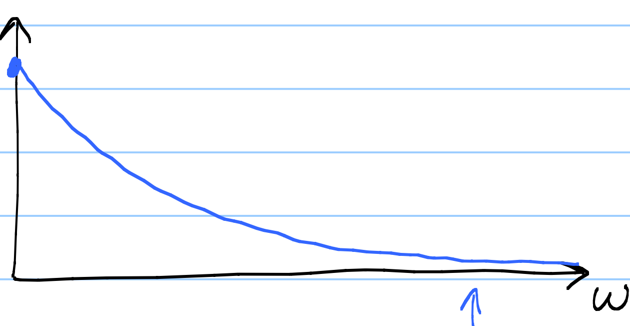
exactly one min, $\omega > 0$.

Freshman calc: Min happens where deriv vanishes:

$$\frac{d}{d\omega}\left(m^2 (\omega_0^2 - \omega^2)^2 + c\omega^2 \right) = 0$$

Get $\omega_{max} = \omega_0 \sqrt{1 - \frac{c^2}{2mK}}$ $\leftarrow$ if $1 - \frac{c^2}{2mK} < 0$, max happens at $\omega = 0$.

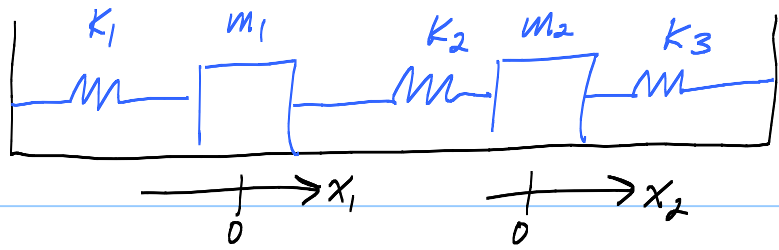
If $1 - \frac{c^2}{2mK} > 0$, then ω_{max} is freq of "practical resonance."



Amp of response

Goes to zero as $\omega \to \infty$.

Prob:



no friction

(slack at 0)

For m_1 :

$$\begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) \end{cases}$$

A system of two second order eqns!

Assume $k_1 = k_2 = k_3 = \underline{1} = m_1 = m_2$

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases} \leftarrow \boxed{x_2 = \ddot{x}_1 + 2x_1}$$

Hmmm. So

$$\boxed{\ddot{x}_2 = x_1^{(4)} + 2\ddot{x}_1}$$

$$\left(x_1^{(4)} + 2\ddot{x}_1\right) = x_1 - 2\left(\ddot{x}_1 + 2x_1\right)$$

plug boxes into 2nd eqn

"Method of elimination"

$$\boxed{x_1^{(4)} + 4\ddot{x}_1 + 3x_1 = 0}$$

$$r^4 + 4r^2 + 3 = 0$$

$$(r^2+3)(r^2+1) = 0$$

$$r = \pm i, \pm\sqrt{3}\, i$$

So $\boxed{x_1 = c_1 \cos t + c_2 \sin t + c_3 \cos\sqrt{3}\,t + c_4 \sin\sqrt{3}\,t}$

Hmmm: $x_2 = \ddot{x}_1 + 2x_1$ (box #1)

$$= \left[-c_1 \cos t - c_2 \sin t - 3c_3 \cos\sqrt{3}t - 3c_4 \sin\sqrt{3}t\right]$$

$$+ 2c_1 \cos t + 2c_2 \sin t + 2c_3 \cos\sqrt{3}t + 2c_4 \sin\sqrt{3}t$$

$$\boxed{x_2 = c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}t - c_4 \sin\sqrt{3}t}$$

To pin down 4 c's, need 4 init cond:

$x_1(0) = A_1$ $\qquad \dot{x}_1(0) = B_1$

$x_2(0) = A_2$ $\qquad \dot{x}_2(0) = B_2$

Do these pin down c's?

Yes! "Wronskian" = det $\begin{bmatrix} \cos t & \sin t & \cos\sqrt{3}t & \sin\sqrt{3}t \\ & & & \\ & & & \end{bmatrix}$

is $\neq 0$.

What are the "basic sol^{n}s" ← modes of oscillation

$c_1 = 1$, other c's $= 0$: $\begin{cases} x_1 = \cos t \\ x_2 = \cos t \end{cases}$ (middle spring not having an effect)

$c_2 = 1$, other c's $= 0$: $\begin{cases} x_1 = \sin t \\ x_2 = \sin t \end{cases}$ (same)

$c_3 = 1$, others $= 0$

$$\begin{cases} x_1 = \cos\sqrt{3}\,t \\ x_2 = -\cos\sqrt{3}\,t \end{cases}$$

faster, and opposing!

$c_4 = 1$, others $= 0$

$$\begin{cases} x_1 = \sin\sqrt{3}\,t \\ x_2 = -\sin\sqrt{3}\,t \end{cases}$$

same

All motions given by a sum of 4 basic modes.

<u>Example:</u> $x_1(0) = 0 \qquad x_2(0) = 1$

$\dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$

Get

$$\begin{cases} x_1 = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{3}\,t = \sin\left(\frac{\sqrt{3}-1}{2}t\right)\sin\left(\frac{\sqrt{3}+1}{2}t\right) \\ x_2 = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{3}\,t = \underbrace{\cos\left(\frac{\sqrt{3}-1}{2}t\right)}_{\text{slow wiggler}}\underbrace{\cos\left(\frac{\sqrt{3}+1}{2}t\right)}_{\text{fast wiggler}} \end{cases}$$