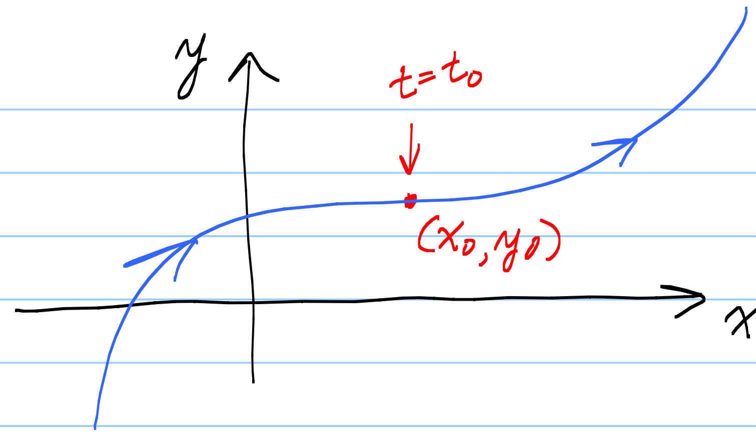
Lecture 23 — 4.2 Systems

HW 22 due in MyLab

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,t) \\[2ex] \dfrac{dy}{dt} = g(x,y,t) \end{cases}$$



IVP: $\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$

E & U Thm: f, g and all their first partials are continuous.
Then solⁿ to IVP exists on some interval $(t_0-\delta, t_0+\delta)$
and the solⁿ is unique.

Linear systems:
$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)x + p_{12}(t)\,y \\[2ex] \dfrac{dy}{dt} = p_{21}(t)x + p_{22}(t)\,y \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

Improved E & U Thm: Solⁿ exists and is unique
on the largest interval where all the p's are continuous.

EX: Linear first order 2x2 homogeneous system with constant coeff.

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 & (1) \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 & (2) \end{cases}$$

Notice: $x_1 \equiv 0$, $x_2 \equiv 0$ solves system.

(The origin is a "critical point.")

Method of Elimination: Solve (1) for x_2.

Plug it into (2). Get 2nd order linear const coeff homog eqn in x_1. Solve. Next, get x_2.

Write
$$\begin{cases} x_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ x_2 = c_1 A e^{r_1 t} + c_2 B e^{r_2 t} \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ A \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} 1 \\ B \end{pmatrix} e^{r_2 t}$$

Preview: r's are eigenvalues of $\mathbb{A}$, vectors are corresponding eigenvectors.

Lazy notation: $\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \lim_{t \to t_0} \begin{pmatrix} DQ\, x_1 \\ DQ\, x_2 \end{pmatrix}$

$$\begin{cases} x_1' = 5x_1 - x_2 \\ x_2' = 3x_1 + x_2 \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x} = \mathbb{A} \vec{x}$$

Big idea: Try sol'ns of form $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{rt}$ $\quad \begin{cases} x_1 = a_1 e^{rt} \\ x_2 = a_2 e^{rt} \end{cases}$

$$\vec{x} = \vec{a} e^{rt}$$

Want

$$\begin{cases} \dfrac{d}{dt}(a_1 e^{rt}) = 5(a_1 e^{rt}) - (a_2 e^{rt}) \\ \dfrac{d}{dt}(a_2 e^{rt}) = 3(a_1 e^{rt}) + (a_2 e^{rt}) \end{cases}$$

$$(\vec{a} e^{rt})' = \mathbb{A}\vec{a} e^{rt}$$

$$\begin{cases} r a_1 e^{rt} = 5a_1 e^{rt} - a_2 e^{rt} \\ r a_2 e^{rt} = 3a_1 e^{rt} + a_2 e^{rt} \end{cases} \longleftarrow r\vec{a} e^{rt} = \mathbb{A}\vec{a} e^{rt}$$

$$\begin{cases} r a_1 = 5a_1 - a_2 \\ r a_2 = 3a_1 + a_2 \end{cases}$$

$$\boxed{r\vec{a} = \mathbb{A}\vec{a}}$$

$\vec{a}$ eigenvector, r eigenvalue for $\vec{a}$ $(\vec{a} \neq 0)$

Don't want $a_1 = 0$, $a_2 = 0$ only

$$(*) \begin{cases} 0 = (5-r)a_1 - a_2 \\ 0 = 3a_1 + (1-r)a_2 \end{cases}$$

$$\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} \vec{a} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$$

Hmmm. Cramer's rule

$$a_1 = \frac{\det \begin{bmatrix} 0 & -1 \\ 0 & 1-r \end{bmatrix}}{\det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix}} = 0$$

$a_2 = 0$ too! Ouch!

We need Cramer's rule to bomb!

Need

$$\det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = 0$$

$$\det(\mathbb{A} - r\mathbb{I}) = 0$$

Need $(5-r)(1-r)-(-1)3=0$

$$r^2-6r+8=0$$

$$(r-4)(r-2)=0$$

$r=2,4$ eigenvalues.

Next, find $\vec{a}$'s for each r.

For $r=2$: (Plug $r=2$ in (*))

$$\begin{cases}(5-2)a_1 - a_2 = 0\\ 3a_1 + (1-2)a_2 = 0\end{cases}$$

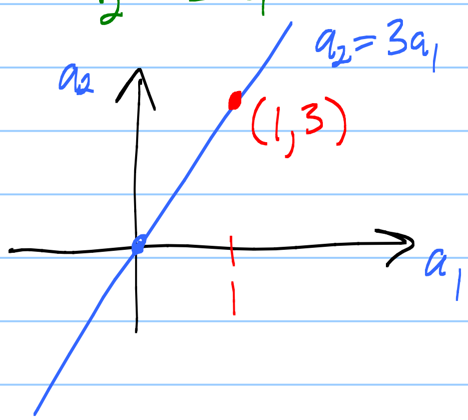
$$\begin{cases}3a_1 - a_2 = 0\\ 3a_1 - a_2 = 0 \quad !\end{cases}$$

Only one eqn! (Cramer's bombs)

$$\boxed{(A-rI)\vec{a} = \vec{0}}$$

$$(A-2I)\vec{a} = \vec{0}$$

$a_2 = 3a_1$



Just want one nonzero $\vec{a}$.

Pick a nice one $\vec{a}=\begin{pmatrix}1\\3\end{pmatrix}$

Get sol^n $\vec{x} = \begin{pmatrix}1\\3\end{pmatrix}e^{2t}$

For $r=4$: Get $\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$

Superposition principle: If $\vec{x}_1$ and $\vec{x}_2$ solve a linear homog system, so do linear combo's!

Big question: Is $c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$ then gen^l sol^n? Can I solve any and all IVPs with it?

$$\begin{cases} x_1(0) = A \\ x_2(0) = B \end{cases} \qquad c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2\cdot 0} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4\cdot 0} \overset{\text{want}}{=} \begin{pmatrix}A\\B\end{pmatrix}$$

Hmmm. Wronskian?

$$\underbrace{\begin{bmatrix}1 & 1\\3 & 1\end{bmatrix}}_{\det = 1-3 = -2 \neq 0}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$$

Cramer's works!

Next time: What if $r_1 = r_2$ or r's are complex?