Lecture 24 5.1 Systems & linear algebra

HW 23 due in MyLab
HW20w due in GS

$f_1(t), f_2(t)$ non-homog.

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - 3x_2 \quad + \\[2ex] \dfrac{dx_2}{dt} = -2x_1 + 2x_2 \quad + \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{x}\,' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\mathbb{A} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

$$\vec{x}\,' = \mathbb{A}\vec{x}$$

Try $\vec{x} = \vec{a}e^{rt}$: $\quad (\vec{a}e^{rt})' \overset{\text{want}}{=} \mathbb{A}\vec{a}e^{rt}$

$$r\vec{a}e^{rt} = \mathbb{A}\vec{a}e^{rt} \qquad \text{(Cancel out } e^{rt}\text{'s)}$$

$$\boxed{r\vec{a} = \mathbb{A}\vec{a}} \qquad (\vec{a} \neq 0)$$

r eigenvalue
$\vec{a}$ eigenvector for r.

$$\mathbb{A}\vec{a} - r\vec{a} = \vec{0}$$

$$\boxed{(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}} \leftarrow \text{for } \vec{a} \neq 0, \text{ must be } \det = 0.$$

Step 1: Find e.vals;

$$\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0} \leftarrow \text{Char. equ.}$$

$$\det \begin{bmatrix} 1-r & -3 \\ -2 & 2-r \end{bmatrix} = 0$$

$$(1-r)(2-r) - (-2)(-3) = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

E. vals: $r = -1, 4$

Step 2: For $r=-1$, find e. vect. $\vec{a}$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$$

plug $r=-1$

$$\begin{bmatrix} 1-(-1) & -3 \\ -2 & 2-(-1) \end{bmatrix} \vec{a} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right]$$ $\leftarrow$ $2a_1 - 3a_3 = 0$ $\leftarrow$ expect this!

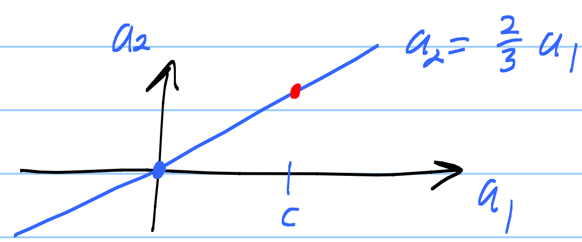
$$\sim \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Nice trick: Spot easy non-zero $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right]$$ $\vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix}$ or $\begin{pmatrix} -B \\ A \end{pmatrix}$

or: $2a_1 - 3a_2 = 0$

$$a_2 = \frac{2}{3}a_1$$



$a_1 = c$

$a_2 = \frac{2}{3}c$

$$\vec{a} = \begin{pmatrix} c \\ \frac{2}{3}c \end{pmatrix}, \quad c \neq 0$$

$\vec{a} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} c$, a whole line of choices. $(c \neq 0)$

Pick a nice one! Take $c = 3$. $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ✓

Get sol^n $\vec{x}_1 = \vec{a} e^{rt} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}$

$$\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

For $r = 4$: $(A - 4I)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 1-4 & -3 & 0 \\ -2 & 2-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -2 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \quad -\frac{1}{3}R1$$

$$\sim \left[\begin{array}{cc|c} \underset{\uparrow A}{1} & \underset{\uparrow B}{1} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R2 + 2R1$$

Trick: $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} B \\ -A \end{pmatrix}$ $\left(\text{or } \begin{pmatrix} -B \\ A \end{pmatrix} \right)$

Get $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

Superposition princ-

$$\vec{x} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$ is a sol^n.

Big ques: Is it all sol^{n}s? The gen^l sol^n

Equiv ques (because of E & U Thm) is, can we solve any and all IVP's?

want

$$\vec{x}(t_0) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t_0} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t_0} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{cases} 3c_1 e^{-t_0} + c_2 e^{4t_0} = A \\ 2c_1 e^{-t_0} - c_2 e^{4t_0} = B \end{cases}$$

$$\begin{bmatrix} 3e^{-t_0} & e^{4t_0} \\ 2e^{-t_0} & -e^{4t_0} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$\vec{x}_1(t_0)$ $\quad$ $\vec{x}_2(t_0)$

Aha! Need

$$\det \begin{bmatrix} 3e^{-t_0} & e^{4t_0} \\ 2e^{-t_0} & -e^{4t_0} \end{bmatrix} \neq 0$$

$$e^{-t_0} \cdot e^{4t_0} \cdot \det \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$$

Never zero

$$e^{3t_0} \left(3 \cdot (-1) - 1 \cdot 2 \right) = -5 e^{3t_0}$$

Wronskian test: Need $\underbrace{\det[\vec{x}_1, \vec{x}_2]}_{\text{Wronskian } = W[\vec{x}_1, \vec{x}_2]} \neq 0$

Abel's: Either $W \equiv 0$ or W is never zero.

Enough to check at only one t_0.

Fact: $W[\vec{x}_1, \vec{x}_2] \neq 0$ equiv to:

$\vec{x}_1$ and $\vec{x}_2$ are lin. indep. sol^{n}s to system.

Linear alg: cA

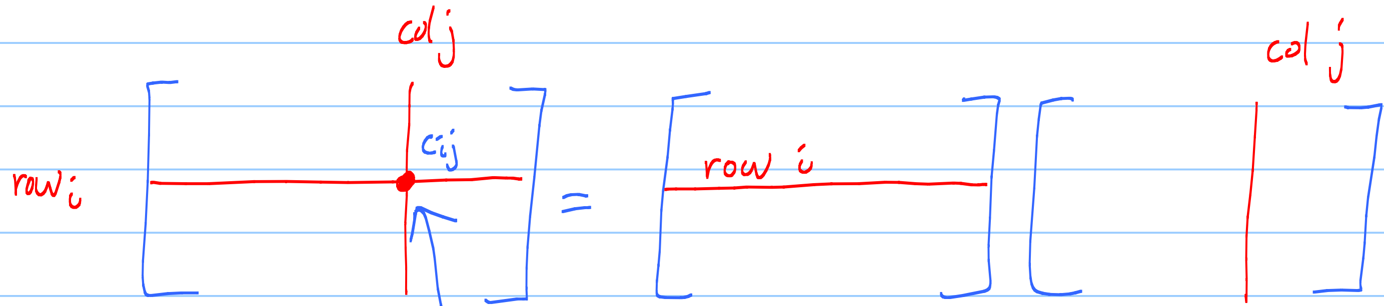
$$10\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\underbrace{A + B}_{\text{same size}}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \text{col } 1, & \text{col } 2, & \ldots, & \text{col } n \end{bmatrix}$$



$C = AB$ (row i of A) $\cdot$ (col j of B)

why?! $\mathbb{A}\vec{x} = \vec{b} \leftarrow$ lot's of $\vec{b}$s.

$$\mathbb{A}\vec{x}_1 = \vec{b}_1, \quad \mathbb{A}\vec{x}_2 = \vec{b}_2, \ldots$$

$$\left[\mathbb{A} \;\middle|\; \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right] = \left[\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\right]$$

Do all at same time!

<u>Inverse of a matrix</u>: Cramer's rule!

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbb{A}} \vec{x} = \vec{A}$$

$$\vec{x} = \underbrace{\frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\mathbb{A}^{-1}} \vec{A}$$

<u>Warning</u>: $\mathbb{AB} \neq \mathbb{BA}$ <u>often</u> for matrices.

$$\mathbb{A}\vec{x} = \vec{b} \leftarrow \text{mult. on left by } \mathbb{A}^{-1}$$

$$\underbrace{\mathbb{A}^{-1}\mathbb{A}}_{\mathbb{I}}\vec{x} = \mathbb{A}^{-1}\vec{b}$$

$$\underbrace{\mathbb{I}\vec{x}}_{\vec{x}} = \mathbb{A}^{-1}\vec{b}$$