Lecture 25 5.2 Linear homog systems with constant coeff HW 24 due in MyLab

$$\underset{n\times k}{C} = \underset{n\times m}{A}\,\underset{m\times k}{B}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 6 & 12 \\ 15 & 30 \end{bmatrix}}$$

Cramer's rule: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$

$$\begin{cases} x_1 = \dfrac{\det\begin{pmatrix} A & b \\ B & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \dfrac{Ad - bB}{\det} \\[2em] x_2 = \dfrac{\det\begin{pmatrix} a & A \\ c & B \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \dfrac{-Ac + aB}{\det} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

switch diag elem

Take minus of off diag terms

Fact: $AA^{-1} = A^{-1}A = I$ (A square)

A^{-1} exists exactly when $\det(A) \neq 0$.

Complex e. vals:

EX: $\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases}$

$$\vec{x}' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{x}$$

Try $\vec{x} = \vec{a}e^{rt}$: Need $A\vec{a} = r\vec{a}$, $\vec{a} \neq 0$

$$\boxed{(A - rI)\vec{a} = \vec{0}} \quad \#2$$

For $\vec{a} \neq 0$, need $\boxed{\det(A - rI) = 0}$ #1

$$\det \begin{bmatrix} -2-r & 2 \\ -2 & -2-r \end{bmatrix} = (-2-r)(-2-r) - (2)(-2)$$

$$= (r+2)^2 + 4 = 0$$

$$(r+2)^2 = -4$$

$$r+2 = \pm 2i$$

$$\boxed{r = -2 \pm 2i}$$

pretend not to notice!

For $r = -2 + 2i$:

$$\begin{bmatrix} -2-(-2+2i) & 2 \\ -2 & -2-(-2+2i) \end{bmatrix} \vec{a} = \vec{0}$$

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \leftarrow R2 = -i \cdot R1 \text{ expected!}$$

$$\sim \left[\begin{array}{cc|c} -2 & -2i & 0 \\ -2i & 2 & 0 \end{array}\right] \leftarrow \text{swap}$$

$$\sim \left[\begin{array}{cc|c} -2 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow R2 - iR1$$

$$\sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow -\tfrac{1}{2} R1$$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix} \text{ or } \begin{pmatrix} -B \\ A \end{pmatrix}$$

Lazy way.

$$a_1 + i a_2 = 0$$

$$a_2 = \frac{1}{i}(-a_1) = i a_1$$

Let $a_1 = c$. Then $a_2 = ic$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} c \\ ic \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} c \quad , \quad c \neq 0$$

Take $c = 1$. Get $\vec{a} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ ← complex e.vect for $r = -2+2i$

Hmmm. $\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+2i)t}$ is a complex sol^n!

Fact: Real and imag parts of <u>one</u> complex sol^n are <u>real</u> sol^{n}s. (Get two from one!)

How to get : $\begin{pmatrix}1\\i\end{pmatrix}e^{(-2+2i)t} = \underline{\underline{\vec{u}}} + i\,\underline{\underline{\vec{v}}}$

$\vec{u}, \vec{v}$ real sol^{n}s.

Step 1: Split $\vec{a} = \begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}$ into real and imag vectors

Step 2: Split $e^{(-2+2i)t} = e^{-2t + i(2t)}$

$$= e^{-2t}e^{i(2t)}$$

$$= e^{-2t}(\cos 2t + i \sin 2t)$$

Step 3: Multiply:

$$\left[\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right]\left(e^{-2t}\cos 2t + i e^{-2t}\sin 2t\right)$$

$$\underbrace{\left[\begin{pmatrix}1\\0\end{pmatrix}e^{-2t}\cos 2t \underset{\uparrow\, i^2}{-} \begin{pmatrix}0\\1\end{pmatrix}e^{-2t}\sin 2t\right]}_{\text{one real sol}^n\ \vec{u}} + i\underbrace{\left[\begin{pmatrix}1\\0\end{pmatrix}e^{-2t}\sin 2t + \begin{pmatrix}0\\1\end{pmatrix}e^{-2t}\cos 2t\right]}_{\text{another } \vec{v}}$$

Do these yield a gen^l sol^n? Are they linearly indep?

Easier to check Wronskian. Is $W \neq 0$?

Is $W \neq 0$ at $t=0$?

$$\vec{x} = c_1 \underbrace{\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix}}_{\vec{x}_1} e^{-2t} + c_2 \underbrace{\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}}_{x_2} e^{-2t}$$

$$W = \det\left[\vec{x}_1 \,,\; \vec{x}_2\right]$$

$$W = \det\left[\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix}e^{-2t} \,,\; \begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}e^{-2t}\right]$$

$$= e^{-2t}\cdot e^{-2t}\cdot \underbrace{\det\begin{bmatrix}\cos & \sin\\ -\sin & \cos\end{bmatrix}}_{\cos^2+\sin^2 \equiv 1} = e^{-4t}$$ never zero!

$$W(0) = \det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1.$$ ← Lazy way via Abel's:

Why it works: Supperposition princ:

Two sol^{n}s $\vec{x}_1, \vec{x}_2$ to $\vec{x}' = A\vec{x}$

$$(c_1\vec{x}_1 + c_2\vec{x}_2)' \stackrel{?}{=} A(c_1\vec{x}_1 + c_2\vec{x}_2)$$

$$\underline{\underline{c_1\vec{x}_1'}} + \underline{\underline{c_2\vec{x}_2'}} \stackrel{?}{=} \underline{\underline{c_1 A\vec{x}_1}} + \underline{\underline{c_2 A\vec{x}_2}}$$

Aha! $(\vec{u} + i\vec{v})' = A(\vec{u} + i\vec{v})$ ← one complex sol^n

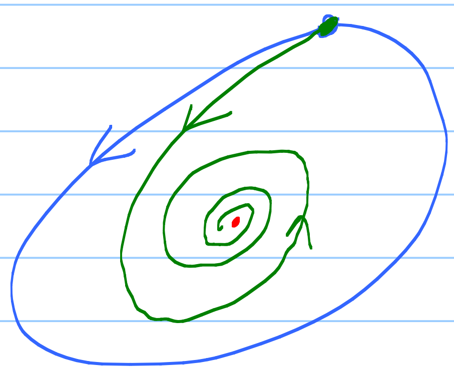
$$\underline{\underline{\vec{u}'}} + i\,\underline{\underline{\vec{v}'}} = \underline{\underline{A\vec{u}}} + i\,\underline{\underline{A\vec{v}}}$$

real parts = imag parts = too!

$\vec{u}, \vec{v}$ two real sol^{n}s.

Cool thing: $c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$

$$\underbrace{e^{-2t}}_{\text{shrinks away!}} \begin{pmatrix} \text{vector of} \\ \underbrace{\text{sin's cos's}}_{\text{periodic!}}. \end{pmatrix}$$



periodic

$(\text{periodic}) \cdot e^{-2t}$

• = origin

Spiral in! Asymptotically stable "sink"

Fun ques: Clockwise or counterclockwise?