Lecture 26 5.2, 5.5 Complex & repeated eigenvalues

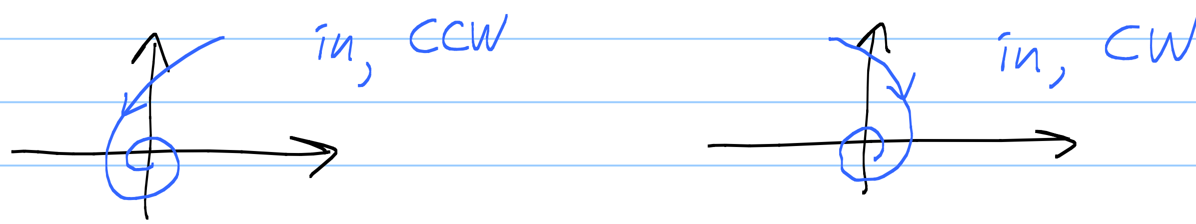
HW 25 due in MyLab

Last time
$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases} \qquad \vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{x}$$

Roots of $\det(A - rI) = 0$ were $r = -2 \pm 2i$

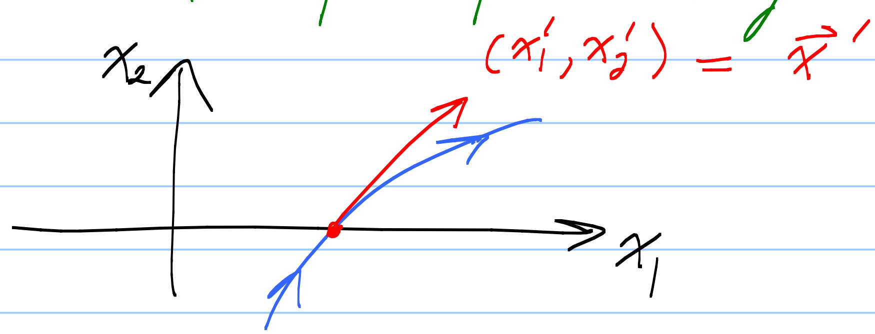
Got $\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$ ← spirals in

Facts: $(0,0)$ is a critical pt. $\begin{cases} x_1 \equiv 0 \\ x_2 \equiv 0 \end{cases}$ is a const sol'n.

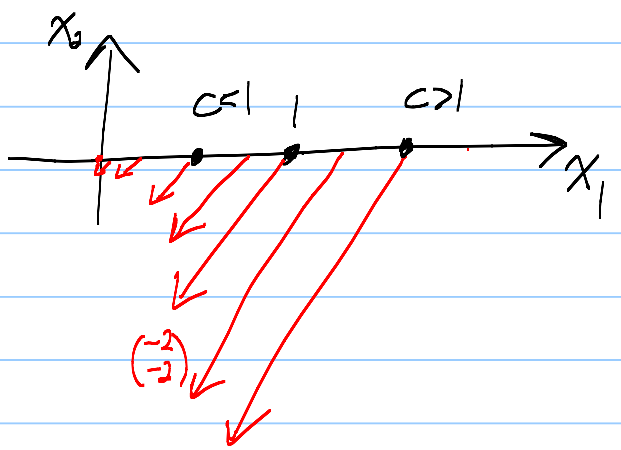


Clockwise (CW) or Counterclockwise (CCW)?

Idea: Test the direction field in the phase plane along the positive real axis.



$\vec{x}\,' = A\vec{x}$



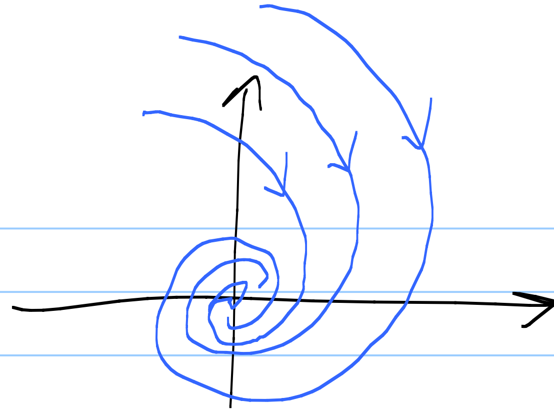
Test field at $\vec{x} = \binom{c}{0}$:

$$\vec{x}\,'\Big|_{\binom{c}{0}} = A\binom{c}{0} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\binom{c}{0}$$

<0 CW
>0 CCW

$$= \begin{pmatrix} -2c \\ -2c \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} c$$

Aha! Clockwise!



In v.s. out?

EX: Complex sol^n: $\begin{pmatrix} 3+4i \\ 5+6i \end{pmatrix} e^{(2+3i)t}$

$$\left[\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \end{pmatrix} i\right] \left(e^{2t} \cos 3t + i e^{2t} \sin 3t \right)$$

$$\left[\begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{2t} \cos 3t \underset{\substack{\uparrow \\ i^2}}{-} \begin{pmatrix} 4 \\ 6 \end{pmatrix} e^{2t} \sin 3t \right] + i \left[\begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{2t} \sin 3t + \begin{pmatrix} 4 \\ 6 \end{pmatrix} e^{2t} \cos 3t \right]$$

$\vec{x}_1$ real sol^n $\qquad$ $\vec{x}_2$ a 2nd real sol^n

Spiral out because of e^{2t} part.

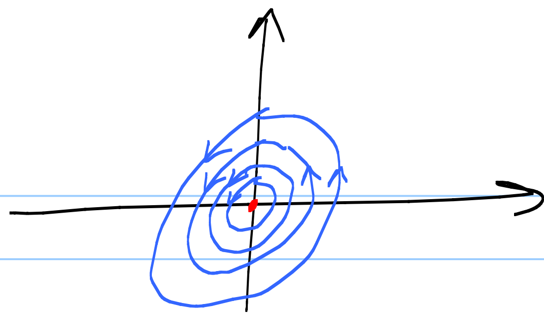
Fact: Complex e.vals: $r = a \pm bi$

$a > 0$ Spirals out. (Unstable "source")

$a < 0$ Spirals in. (Asymt stable "sink")

$a = 0$? "Center", stable (but not asymt. stable)

Fact: Trajectories in phase plane are ellipses (center).



Formula: $(\vec{A}+\vec{B}i)(e^{at}\cos bt + i e^{at}\sin bt)$

$$\underbrace{\left[\vec{A}e^{at}\cos bt \underset{\uparrow\, i^2}{-} \vec{B}e^{at}\sin bt\right]}_{\vec{x}_1} + i\underbrace{\left[\vec{A}e^{at}\sin bt + \vec{B}e^{at}\cos bt\right]}_{\vec{x}_2}$$

3x3 case:

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_2 \\[2mm] \dfrac{dx_2}{dt} = -x_1 \\[2mm] \dfrac{dx_3}{dt} = x_1 + 4x_2 - x_3 \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix}\vec{x}$$

$$\det(A - r\mathbb{I}) = 0 \qquad \det\begin{bmatrix} 0-r & 4 & 0 \\ -1 & 0-r & 0 \\ 1 & 4 & -1-r \end{bmatrix} = 0$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$(+1)(-1-r)\det\begin{pmatrix} -r & 4 \\ -1 & -r \end{pmatrix} + (-1)\cdot 0\cdot \det\begin{pmatrix} -r & 4 \\ 1 & 4 \end{pmatrix}$$

$$+(+1)\cdot 0\cdot \det\begin{pmatrix}-1 & -r\\ 1 & 4\end{pmatrix}$$

$$(-1-r)\left[(-r)(-r)-4(-1)\right]=0$$

$$(-1-r)\left(r^2+4\right)=0 \qquad \text{Roots}: \quad r=-1,\ r=\pm 2i$$

For $r=-1$: $\quad (A-rI)\vec{a}=\vec{0}$

$r=-1$

$$\left[\begin{array}{ccc|c} 0-(-1) & 4 & 0 & 0\\ -1 & 0-(-1) & 0 & 0\\ 1 & 4 & -1-(-1) & 0\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0\\ -1 & 1 & 0 & 0\\ 1 & 4 & 0 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0\\ 0 & 5 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \begin{array}{l} R1\\ R2+R1\\ R3-R1\end{array}$$

$$a_1 + 4a_2 = 0 \leftarrow a_1=0 \text{ too!}$$

$$5a_2 = 0 \leftarrow a_2 = 0$$

Hmmm. a_3 can be anything! $\quad a_3 = c.$ $\qquad \begin{pmatrix}0\\0\\c\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}c$

Need $c \neq 0$ to avoid getting zero vector. Take $c = 1$.

Get $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and one sol^n $\vec{x}_1 = \vec{a} e^{rt} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t$

<u>For $r = 2i$:</u> Get a complex e.vect. $\vec{A} + \vec{B}i$

Get complex sol^n $[\vec{A} + \vec{B}i](\cos 2t + i \sin 2t)$

Get <u>two</u> real sol^n's from <u>one</u> complex e. val.

Gen^l Sol^n:
$$\vec{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\cos 2t \\ -\sin 2t \\ 2\cos 2t \end{pmatrix} + c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \\ 2\sin 2t \end{pmatrix}$$

Plan: 5.2, $\underbrace{5.5}_{\text{repeated e.vals}}$, $\underbrace{5.3}_{\text{gallery}}$

<u>EX:</u>
$$\vec{x}\,' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$\det(A - r\mathbb{I}) = \det \begin{bmatrix} 1-r & 0 \\ 0 & 1-r \end{bmatrix} = (1-r)^2 = 0$$

$r = 1, 1$ is repeated.

<u>For $r=1$:</u> $(A - r\mathbb{I})\vec{a} = \vec{0}$

$r=1$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

What?

a_1, a_2 can both be anything!

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2 linearly independent e. vects!

Get $\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ two indep e. vects.

Get $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \qquad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$

$$W(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0.$$

Gen^l sol^n: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$