Lecture 27 5.5 repeated eigenvalues

HW 26 due in MyLab

HW23w, 24w, 25w due in GS

Exam 2 on Tues, Nov. 9, 8:00–9:00 pm in WALC 1055

See home page for practice problems

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow \quad a_1 - a_3 = 0 \quad (1) \\ \leftarrow \quad a_2 - \frac{i}{2} a_3 = 0 \quad (2) \end{array}$$

a_1, a_2 bound vars

a_3 is a free var

Let $a_3 = c$.

(1) $a_1 = a_3 = c$

(2) $a_2 = \frac{i}{2} a_3 = \frac{i}{2} c$

$$\begin{cases} a_1 = c \\ a_2 = \frac{i}{2} c \\ a_3 = c \end{cases} \qquad \vec{a} = \begin{pmatrix} c \\ \frac{i}{2} c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} c, \quad c \neq 0.$$

Take $c=2$; Get complex e.vect $\begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix}$ for $r = 2i$

Complex sol^n $\left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i\right] (\cos 2t + i \sin 2t)$

Get $\vec{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cos 2t \underset{\uparrow\, i^2}{-} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t = \begin{pmatrix} 2\cos 2t \\ -\sin 2t \\ 2\cos 2t \end{pmatrix}$

$\vec{x}_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t = \begin{pmatrix} 2\sin 2t \\ \cos 2t \\ 2\sin 2t \end{pmatrix}$

Repeated roots:

Last time $\vec{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

$$\det\begin{pmatrix} 2-r & 0 \\ 0 & 2-r \end{pmatrix} = 0$$

$$(2-r)^2 = 0 \qquad r = 2, 2$$

$$(\mathbb{A} - 2\mathbb{I})\vec{a} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{a} = 0 \quad ! \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

two lin. indep e.vects for $r=2$

Get <u>two</u> lin indep sol^{n}s $\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t}$ from $r=2$.

Gen^l Sol^n $\vec{x} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t}$

<u>EX</u>: $\vec{x}\,' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\vec{x}$

$$\det\begin{pmatrix} 1-r & -3 \\ 3 & 7-r \end{pmatrix} = (1-r)(7-r) - 3(-3) = 0$$

$$= r^2 - 8r + 16 = 0$$

$$(r-4)^2 = 0$$

$r = 4, 4$ repeated (algebraic multiplicity two)

How many e.vects? If # = mult., get indep. sol^{n}s, gen^l sol^n,

If # < mult, $\mathbb{A}$ is "defective"

<u>For $r=4$</u>: $(\mathbb{A} - 4\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 1-4 & -3 & 0 \\ 3 & 7-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \quad a_1 + a_2 = 0$$

a_1 bound a_2 free

Let $a_2 = c$.

$a_1 = -a_2 = -c$

$$\begin{cases} a_1 = -c \\ a_2 = c \end{cases} \qquad \vec{a} = \begin{pmatrix} -c \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} c \quad , \quad c \neq 0.$$

Take $c = -1$. Get $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Bummer. Only get one sol'n $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

Need $\vec{x}_2$! Hmmm. Try $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{4t}$. Damn!

Better: Try $\boxed{\vec{x}_2 = \vec{a} t e^{rt} + \vec{b} e^{rt}}$

$$\vec{x}_2' = \vec{a} e^{rt} + \vec{a} r t e^{rt} + \vec{b} r e^{rt}$$

$$= r \vec{a} t e^{rt} + (\vec{a} + r\vec{b}) e^{rt}$$

Plug in and force:

$$\vec{x}_2' \overset{\text{want}}{=} A \vec{x}_2$$

$$\underbrace{r\vec{a}te^{rt} + (\vec{a}+r\vec{b})e^{rt}}_{\vec{x}_2'} = \mathbb{A}\underbrace{\left(\vec{a}te^{rt} + \vec{b}e^{rt}\right)}_{\vec{x}_2}$$

$$\vec{0} = \left[\mathbb{A}\vec{a} - r\vec{a}\right]te^{rt} + \left[\mathbb{A}\vec{b} - r\vec{b} - \vec{a}\right]e^{rt}$$

Divide by e^{rt} $$\underbrace{(\mathbb{A}\vec{a} - r\vec{a})}_{\text{need } = 0}t + \underbrace{(\mathbb{A}\vec{b} - r\vec{b} - \vec{a})}_{\text{need } = 0} = \vec{0}$$

because, first let $t=0$. Then let $t=1$.

Aha! $\boxed{\mathbb{A}\vec{a} = r\vec{a}} \leftarrow$ $\vec{a}$ is an e.vect for r!

$$\mathbb{A}\vec{b} - r\vec{b} = \vec{a}$$

$$\boxed{(\mathbb{A} - r\mathbb{I})\vec{b} = \vec{a}}$$

Important fact: When $\mathbb{A}$ is defective, can find a sol^n to this system, guaranteed.

Got $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Get $\vec{b}$: $$(\mathbb{A} - \underset{r=4}{r}\mathbb{I})\vec{b} = \vec{a}$$

$$\left[\begin{array}{cc|c} 1-4 & -3 & 1 \\ 3 & 7-4 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -3 & 1 \\ 3 & 3 & -1 \end{array}\right]$$

Cramer's bombs, <u>but</u>

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

b_1 bound $\qquad$ b_2 free $\qquad$ Let $b_2 = c$.

$$b_1 + c = -1$$

$$b_1 = -1 - c$$

$$\begin{cases} b_1 = -1 - c \\ b_2 = c \end{cases} \qquad \vec{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} c$$

Just need <u>one</u> $\vec{b}$. Pick a nice one! $\quad c = 0$; $\vec{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{4t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t}}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{4t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{4t} \right]$$

$$\begin{cases} x_1 = c_1 e^{4t} + c_2 \left(t e^{4t} - e^{4t} \right) \\ x_2 = -c_1 e^{4t} - c_2 t e^{4t} \end{cases}$$

$$\vec{x}_2 = \begin{pmatrix} t - 1 \\ -1 \end{pmatrix} e^{4t}$$