Lecture 28 5.3 Pictures!

HW 27 due in MyLab
Exam 2 on Tuesday
in WALC 1055, 8-9 pm

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \\[2ex] \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \end{cases}$$

$$\vec{x}\,' = \mathbb{A}\vec{x}$$

Abel's: $W = Ce^{\int (a_{11}+a_{22})\,dt}$

Try $\vec{x} = \vec{a}e^{rt}$: $\vec{x}\,' = r\vec{a}e^{rt} \overset{\text{want}}{=} \mathbb{A}\vec{x} = \mathbb{A}\vec{a}e^{rt}$

$$\boxed{r\vec{a} = \mathbb{A}\vec{a}}$$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = 0$$

Need $\underbrace{\det(\mathbb{A} - r\mathbb{I}) = 0}_{\text{Characteristic equ}}$ for $\vec{a} \neq 0$.

$\mathbb{A}$ (2x2). Quadratic.

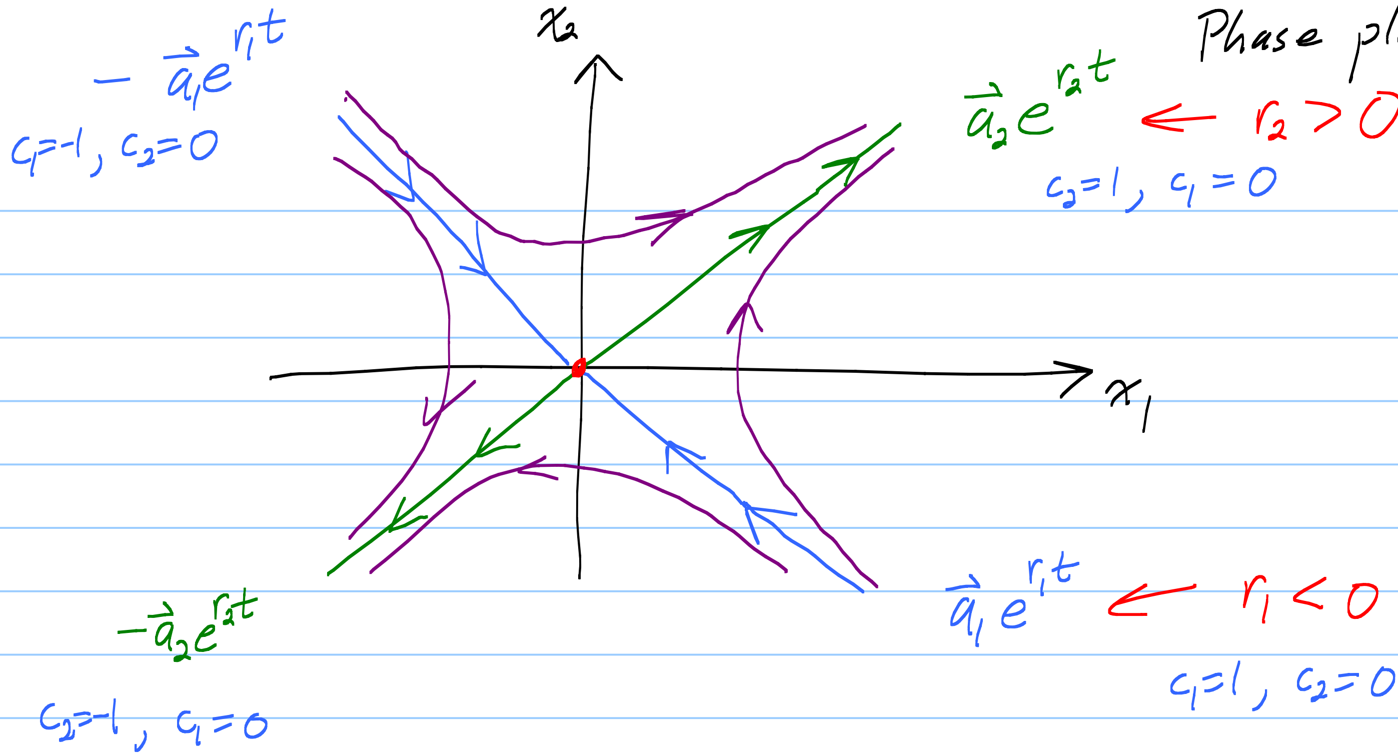
Cases: 1) Roots r_1, r_2 real and opposite sign: $r_1 < 0 < r_2$

Get e.vects $\vec{a}_1$, $\vec{a}_2$ for r_1, r_2, resp.

Fact: e.vects for different e.vals are lin. indep.

Gen' Sol^n: $\vec{x} = c_1\vec{a}_1e^{r_1t} + c_2\vec{a}_2e^{r_2t}$

Phase plane

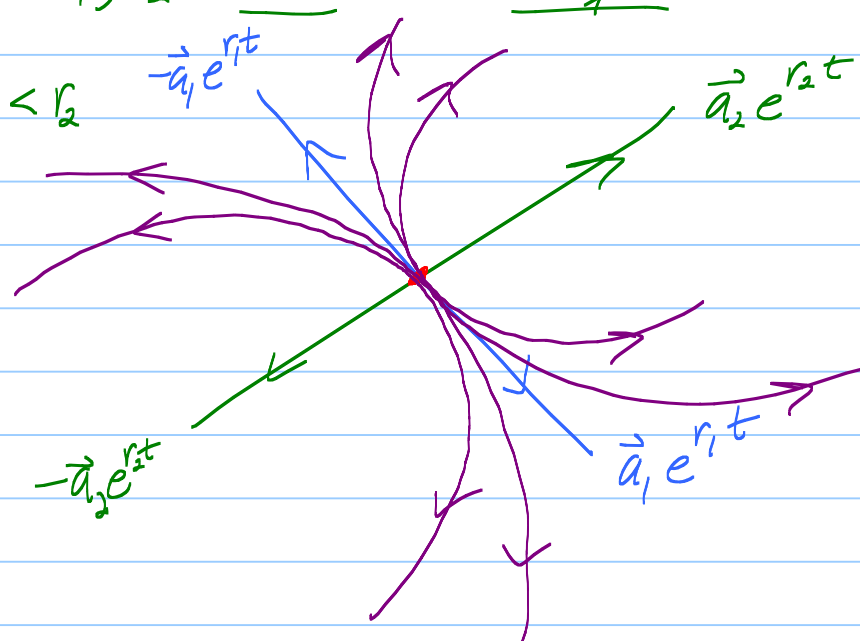


Saddle point. Unstable.

Saddle rule: Trajectories come in asymptotic to $\pm$ e.vects corresponding to the negative e.val and go out asympt to $\pm$e.vectors for positive e.val.

Case: r_1, r_2 real and unequal and same sign.

Subcase: $0 < r_1 < r_2$



Improper node: Unstable. source

Node rule: Near the origin, trajectories hug the $\pm$ e.vect directions corresponding to the e.val closest to zero.

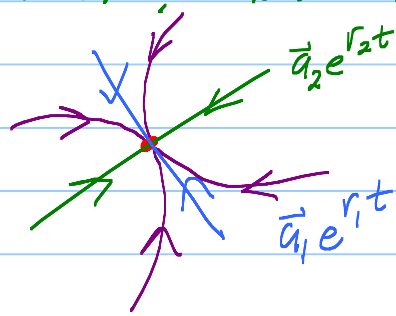
Part 2: Far from origin, traj look parallel to the other e.vect direction

Subcase: $r_2 < r_1 < 0$

r_1 closest to zero

Node rule: Same picture! But all arrows reversed.

Improper node:

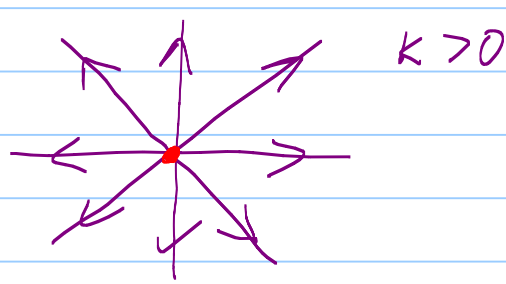


Asympt stable sink.

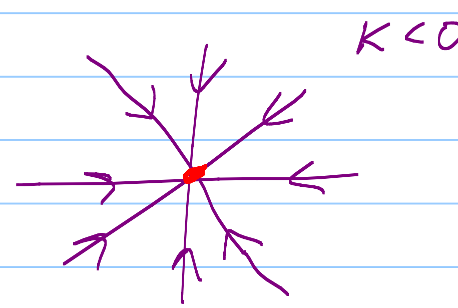
EX: A proper node:

$$\begin{cases} \dfrac{dx_1}{dt} = k\, x_1 \\[2ex] \dfrac{dx_2}{dt} = k\, x_2 \end{cases}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{kt} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{kt} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{kt}$$



Unstable. Source



A. stable sink.

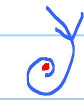
Case: $r = a \pm bi$

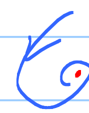
For $r = a+bi$: get complex e.vect $\vec{A}+\vec{B}i$

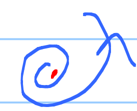
Complex sol^n $\left[\vec{A}+\vec{B}i\right]\left(e^{at}\cos bt + i\, e^{at}\sin bt\right)$

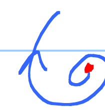
$$= \underbrace{\left[\vec{A}e^{at}\cos bt \underset{i^2}{-} \vec{B}e^{at}\sin bt\right]}_{\vec{x}_1} + i \underbrace{\left[\vec{A}e^{at}\sin bt + \vec{B}e^{at}\cos bt\right]}_{\vec{x}_2}$$

Critical thing: Sign of $a = \mathrm{Re}\, r$

Subcase $a<0$: Type: Spiral (in) sink 

Stability: Asympt. stable 

Subcase $a>0$: Spiral (out) source 

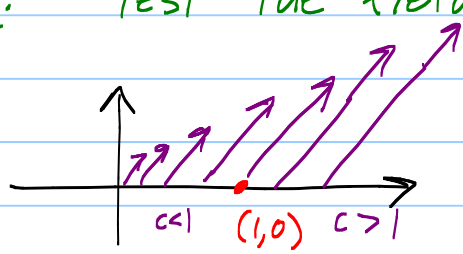
Unstable 

Subcase $a=0$: **Center**

Stability: Stable.

CW or CCW? Test the field at $\begin{pmatrix}1\\0\end{pmatrix} \sim (1,0)$

CCW



$$\vec{x}'\Big|_{\begin{pmatrix}1\\0\end{pmatrix}} = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}$$

← col 1 of A

$$\vec{x}'\Big|_{\binom{c}{0}} = \mathbb{A}\binom{c}{0} = c\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

Up or down direction? $a_{21} > 0$ up. CCW

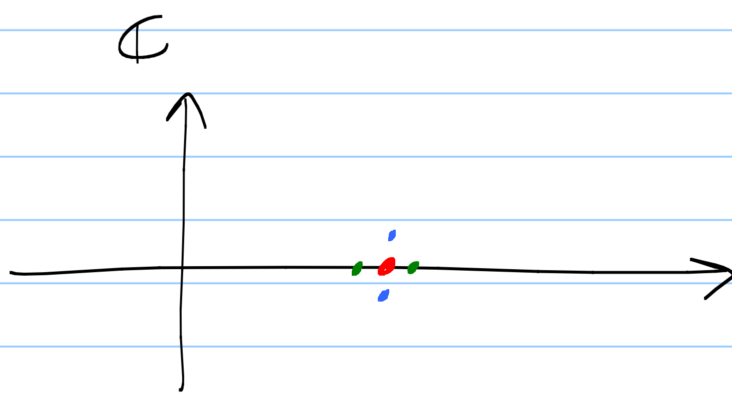
$a_{21} < 0$ down. CW

Spiral rule: Sign of a_{21} determines CW or CCW.

What about $r_1 = r_2$ cases? or r's $= 0$ case.

Very delicate. $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\boxed{b^2 - 4ac = 0}$

$r_1, r_2 = \dfrac{-b}{2a}$



red dot $\dfrac{-b}{2a}$

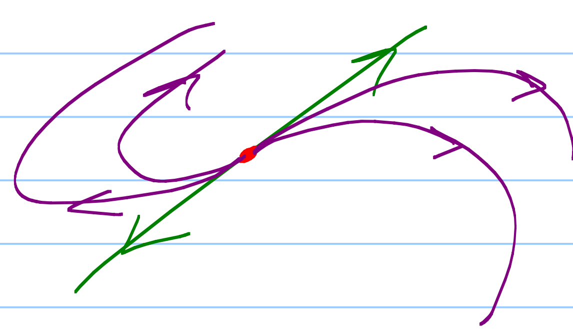
r_1, r_2 $\underbrace{b^2 - 4ac}_{\text{small}} > 0$

r_1, r_2 $\underbrace{b^2 - 4ac}_{\text{small}} < 0$

Typical pictures.

$\mathbb{A}$ non-defective: Proper node.

$\mathbb{A}$ defective:



<u>Interesting fact</u> $\frac{-b}{2a} > 0$ Unstable source

$\frac{-b}{2a} < 0$ A. Stable sink