Lecture 29 The exponential matrix & fundamental matrices

$\underbrace{\text{HW 28 due in MyLab}}_{\text{tonight}}$; HW 26w, 27w, 28w due in GS Thursday, 11:59 pm

Fundamental matrices: $\vec{x}' = \mathbb{A}\vec{x}$, $\vec{x}(0) = \vec{x}_0$, $\mathbb{A}$ $(n \times n)$

$$\begin{cases} \vec{x}_1' = \mathbb{A}\vec{x}_1 \\ \vec{x}_2' = \mathbb{A}\vec{x}_2 \\ \vdots \\ \vec{x}_n' = \mathbb{A}\vec{x}_n \end{cases}$$

need n lin indep sol^{n}s

$$\left[\vec{x}_1', \vec{x}_2', \ldots, \vec{x}_n'\right] = \left[\mathbb{A}\vec{x}_1, \mathbb{A}\vec{x}_2, \ldots, \mathbb{A}\vec{x}_n\right]$$

$$= \mathbb{A}\underbrace{\left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right]}_{\mathbb{X} \quad \text{"fundamental matrix"}}$$

Aha! $\boxed{\mathbb{X}' = \mathbb{A}\,\mathbb{X}}$ ← Hmmm: $y' = ky$ $\quad y = Ce^{kt}$

Gen^l sol^n: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$

$$= \left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right]\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbb{X}\,\vec{c}$$

Solve IVP: $\vec{x}(0) = \vec{x}_0$. Prob: $\underline{\vec{x}(0) = \mathbb{X}(0)\,\vec{c} = \vec{x}_0}$ ← want

Aha! $\mathbb{X}(0)\,\vec{c} = \vec{x}_0$

Multiply on left by $\mathbb{X}(0)^{-1}$:

$$\underbrace{\mathbb{X}(0)^{-1}\mathbb{X}(0)}_{\mathbb{I}}\,\vec{c} = \mathbb{X}(0)^{-1}\vec{x}_0$$

$$\boxed{\vec{c} = \mathbb{X}(0)^{-1}\vec{x}_0}$$

EX: $\mathbb{A}$ (2x2). Gen^l Sol^n is $c_1\begin{pmatrix}5\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$

$$\mathbb{X} = \left[\begin{pmatrix}5\\1\end{pmatrix}e^{3t},\ \begin{pmatrix}1\\-1\end{pmatrix}e^{t}\right] = \begin{bmatrix}5e^{3t} & e^{t}\\ e^{3t} & -e^{t}\end{bmatrix}$$

$$\mathbb{X}(0) = \begin{bmatrix}5 & 1\\ 1 & -1\end{bmatrix}$$

Nice formula: 2x2: $\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{det}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$$\mathbb{X}(0)^{-1} = \frac{1}{5(-1)-1}\begin{bmatrix}-1 & -1\\ -1 & 5\end{bmatrix} = \begin{bmatrix}1/6 & 1/6\\ 1/6 & -5/6\end{bmatrix}$$

Sol^n with $\vec{x}(0) = \begin{pmatrix}6\\7\end{pmatrix}$ is $c_1\begin{pmatrix}5\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$

where $\vec{c} = \mathbb{X}(0)^{-1}\begin{pmatrix}6\\7\end{pmatrix} = \begin{bmatrix}1/6 & 1/6\\ 1/6 & -5/6\end{bmatrix}\begin{pmatrix}6\\7\end{pmatrix} = \begin{pmatrix}13/6\\ -29/6\end{pmatrix}$

$c_1 = 13/6$, $c_2 = -29/6$.

<u>Normalized fundamental matrix</u>: For extra lazy people!

Want a fund matrix with $\mathbb{X}(0) = \mathbb{I}$.

(2x2):

$$\left[\vec{x}_1(0),\ \vec{x}_2(0)\right] = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow\ \ \ \ \uparrow \\ \vec{e}_1\ \ \ \vec{e}_2 \end{array}$$

Aha! Want sol^n's with $\vec{x}_1(0) = \vec{e}_1$, $\vec{x}_2(0) = \vec{e}_2$

Aha! If I get these, they form a gen^l sol^n because

Wronskian $W(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

How to find them: $\vec{x}\,' = \mathbb{A}\vec{x}$

$$\left[\vec{x}_1{}',\ \vec{x}_2{}'\right] = \mathbb{A}\left[\vec{x}_1, \vec{x}_2\right]$$

$$\mathbb{X}' = \mathbb{A}\,\mathbb{X} \quad \leftarrow \text{not the right } \vec{x}\text{'s yet}$$

Want normalized x's : $\vec{\chi}_1, \vec{\chi}_2$

$$\vec{\chi}_1 = \mathbb{X}\vec{c}_1 \quad , \quad \vec{\chi}_2 = \mathbb{X}\vec{c}_2$$

$$\Phi = \left[\chi_1, \chi_2\right] = \left[\mathbb{X}\vec{c}_1, \mathbb{X}\vec{c}_2\right] = \mathbb{X}\left[\vec{c}_1, \vec{c}_2\right]$$

Want

$$\left[\chi_1, \chi_2\right](0) = \mathbb{X}(0)\underbrace{\left[\vec{c}_1, \vec{c}_2\right]}_{\substack{\text{must be} \\ \mathbb{X}(0)^{-1}}} = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \uparrow\ \ \ \ \uparrow \\ \vec{e}_1\ \ \ \vec{e}_2 \end{array}$$

<u>Important formula</u>: Normalized fundamental matrix Φ is given by

$$\boxed{\Phi = \mathbb{X}\,\mathbb{X}(0)^{-1}}$$

Good thing about a Φ. If you have it, sol^n to $\vec{x}(0) = \vec{x}_0$ is just $\Phi \vec{x}_0$. [Before $\vec{x} = X\, X(0)^{-1} \vec{x}_0$]

EX: $X = \begin{bmatrix} 5e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}$ $\quad X(0)^{-1} = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{bmatrix}$

$$\Phi = \begin{bmatrix} 5e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{5}{6}e^{3t} + \frac{1}{6}e^{t} & \frac{5}{6}e^{3t} - \frac{5}{6}e^{t} \\ \frac{1}{6}e^{3t} - \frac{1}{6}e^{t} & \frac{1}{6}e^{3t} + \frac{5}{6}e^{t} \end{bmatrix}$$

$\quad\quad \uparrow \quad\quad\quad\quad \uparrow$

$\quad\quad x_1 \quad\quad\quad\quad x_2 \quad$ sol^{n}s ✓

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Mind blowing math: $\quad y' = ky$

$$y = Ce^{kt}$$

Exponential fcn: e^{kt} solves $\quad \boxed{y' = ky \text{ with } y(0) = 1}$

Normalized fund matrix solves $\quad \boxed{\Phi' = A\Phi \text{ with } \Phi(0) = I}$

Hmmm: $\quad e^{kx} = 1 + \frac{kx}{1!} + \frac{(kx)^2}{2!} + \cdots$

$$\frac{d}{dx}\left(e^{kx}\right) = 0 + k + \frac{2x\,k^2}{2!} + \frac{3x^2 k^3}{3!} + \cdots + \frac{n x^{n-1} k^n}{n!} + \cdots$$

$$= K\left(1 + \frac{kx}{1!} + \frac{(kx)^2}{2!} + \cdots \frac{(kx)^{n-1}}{(n-1)!} + \cdots\right)$$

$$= K e^{kx} \checkmark$$

Hmmm: $e^{\mathbb{A}t} = \mathbb{I} + \frac{1}{1!}\mathbb{A}t + \frac{1}{2!}\mathbb{A}^2 t^2 + \cdots + \frac{1}{n!}\mathbb{A}^n t^n + \cdots$

Can make it work!

Good news: We used diff equs to find

$$\boxed{e^{\mathbb{A}t} = \Phi}$$

Fun facts: Possible for $\mathbb{A}^n = \vec{0}$ (nilpotent)

EX: $\mathbb{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $\mathbb{A}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $\mathbb{A}^3 = \vec{0}$, $\mathbb{A}^4 = \vec{0}$, ...

$$e^{\mathbb{A}t} = \mathbb{I} + \frac{1}{1!}\mathbb{A}t + \underbrace{\frac{1}{2!}\mathbb{A}^2 t^2}_{0} + 0 + 0 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$$

Fun fact #2: Diagonal matrices are nice

$$\mathbb{D} = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$e^{\mathbb{D}t} = \begin{bmatrix} e^{d_1 t} & & 0 \\ & e^{d_2 t} & \\ 0 & & \ddots & \\ & & & e^{d_n t} \end{bmatrix}$$

Fun fact #3: $e^{(\mathbb{A}+\mathbb{B})t} \overset{?}{=} e^{\mathbb{A}t} e^{\mathbb{B}t}$

Yes, if $\mathbb{A}$ and $\mathbb{B}$ commute: $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$

Fact #4: Diagonal matrices comute with any matrix.

HWK prob: $\mathbb{A} = \underset{\text{diag}}{\underset{\uparrow}{\mathbb{D}}} + \underset{\text{nilpotent}}{\underset{\uparrow}{\mathbb{N}}}$

$$e^{\mathbb{A}t} = e^{\mathbb{D}t} e^{\mathbb{N}t}$$