# Exam 2 discussion

*(10 pts.)* The solution to the initial value problem

$$y'' - y' - 2y = -6x + 5, \qquad y(0) = -4,\ y'(0) = 0 \text{ is}$$

$r^2 - r - 2 = 0 \quad (r+1)(r-2) = 0. \quad r = -1, 2 \quad y_c = c_1 e^{-x} + c_2 e^{2x}$

$$y = -e^{2x} + e^{-x} + \underbrace{3x - 4}_{y_p}$$

want

Try $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$0 - A - 2(Ax + B) = -6x + 5$

$\underbrace{-2A}_{-6}x + \underbrace{(-A - 2B)}_{5} = -6x + 5$

$A = 3 \qquad -3 - 2B = 5$

$-2B = 8$

$B = -4$

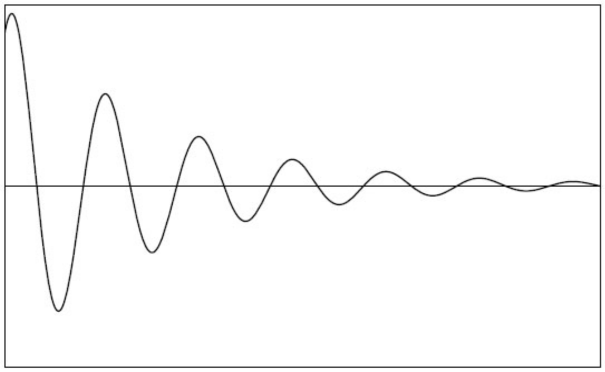
Gen$^l$ Sol$^n$ $\quad y = c_1 e^{-x} + c_2 e^{2x} + 3x - 4$

$y' = -c_1 e^{-x} + 2c_2 e^{2x} + 3$

want

$$\begin{cases} y(0) = c_1 + c_2 - 4 = -4 \\ y'(0) = -c_1 + 2c_2 + 3 = 0 \end{cases}$$

For which values of the parameter $\alpha$ will the equation $y'' + \alpha y' + y = 0$ have solutions whose graphs are similar to the following graph?



$$r^2 + \alpha + 1 = 0$$

$$r = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Must be that

$$\alpha^2 - 4 < 0$$

$$\alpha^2 < 4$$

$$\boxed{|\alpha| < 2}$$

to make wiggle

$$0 < \alpha < 2$$

$$r = \frac{-\alpha}{2} \pm \omega i \quad , \quad \omega = \frac{\sqrt{|\alpha^2 - 4|}}{2}$$

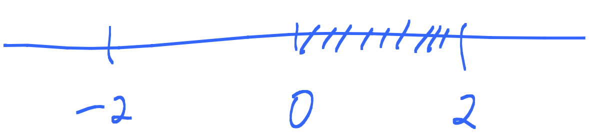
$$y = A e^{-\frac{\alpha}{2} t} \cos(\omega t - \varphi)$$

need $-\frac{\alpha}{2} < 0$ to make it fizzle.

$$\boxed{\alpha > 0}$$

$$-2 < \alpha < 2$$

$$\alpha > 0.$$




$$0 < \alpha < 2$$

(*10 pts.*) Find the general solution of the system

$$\det\begin{pmatrix} 2-r & -1 \\ 1 & 2-r \end{pmatrix} = 0 \qquad \mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\mathbf{x}.$$

$$\mathbf{x} = c_1 e^{2t}\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$(2-r)^2 + 1 = 0$$

$$(2-r) = \pm i$$

$$r = 2 \pm i$$

For $r = 2+i$: $(A - \underset{r=2+i}{r} I)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 2-(2+i) & -1 & 0 \\ 1 & 2-(2+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{Get } \vec{a} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right](\cos t + i \sin t)$$

$$\begin{cases} \vec{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \\ \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \end{cases}$$

*(10 pts.)* If the method of variation of parameters is used to find a particular solution to

$$x^2 y'' - 3xy' + 4y = x^4$$ ← not in standard form!

$F(x) = x^4/x^2 = x^2$

for $x > 0$, given that the general solution to the associated homogeneous equation is $c_1 \underbrace{x^2}_{y_1} + c_2 \underbrace{x^2 \ln x}_{y_2}$, the particular solution obtained is $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{-(x^2 \ln x)(x^2)}{x^3}\, dx \text{ and } u_2 = \int \frac{(x^2)(x^2)}{x^3}\, dx$$ ✓

$$\boxed{u_1' = \frac{-y_2 F(x)}{W}} \qquad \boxed{u_2' = \frac{y_1 F(x)}{W}}$$

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x^2 \cdot \frac{1}{x} \end{bmatrix}$$

$$= \det \begin{bmatrix} x^2 & 0 \\ 2x & x \end{bmatrix} = x^3$$

$C2 - C1(\ln x)$

*(10 pts.)* Find the general solution of the system

$(2-r)(-4-r)+5=0$

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix} \mathbf{x}.$$

$r^2+2r-3=0$

$(r+3)(r-1)=0 \quad r=1,-3$

$$\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $r=1$: $\left[\begin{array}{cc|c} 2-1 & -5 & 0 \\ 1 & -4-1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$a_1$ bound   $a_2$ free

$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{a} = \begin{pmatrix} -B \\ A \end{pmatrix}$ or $\begin{pmatrix} B \\ -A \end{pmatrix}$

$\vec{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ✓   or   $a_1 - 5a_2 = 0$

Let $a_2 = c$.

For $r=-3$ $\left[\begin{array}{cc|c} 2-(-3) & -5 & 0 \\ 1 & -4-(-3) & 0 \end{array}\right]$

Then $a_1 = 5a_2 = 5c$

List: $\begin{cases} a_1 = 5c \\ a_2 = c \end{cases}$   $\vec{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} c$

$c \neq 0.$

$\left[\begin{array}{cc|c} 5 & -5 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{a} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad \frac{1}{5}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ even better!

*(10 pts.)* An undamped, free vibration modeled by $u'' + 64\,u = 0$ has initial conditions $u(0) = 3,\ u'(0) = -32$. The solution of this initial value problem can be written as $u = \underbrace{R\cos(\omega t - \alpha)}$. What is the amplitude $R$ of this solution?

$r^2 + 64 = 0$

$r = \pm 8i$

$R = 5$

$u = c_1 \cos 8t + c_2 \sin 8t$

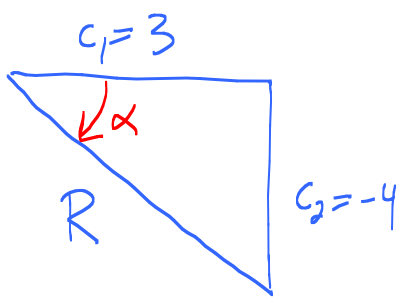
$u' = -8c_1 \sin 8t + 8c_2 \cos 8t$

want

$u(0) = c_1 = 3$

$u'(0) = 8c_2 = -32$

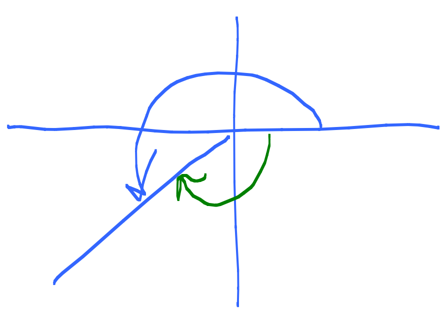
$\boxed{\begin{array}{l} c_1 = 3 \\ c_2 = -4 \end{array}}$




$R = \sqrt{3^2 + 4^2} = 5$ ✓

$\alpha = -\operatorname{Tan}^{-1} \frac{4}{3}$

$u = 5 \cos\left(8t + \operatorname{Tan}^{-1} \frac{4}{3}\right)$



*(10 pts.)* The correct form of a particular solution to use in the method of undetermined coefficients for the equation

$$y^{(4)} + 2y'' + y = t^2 \cos t \quad \text{is}$$

poly of deg 2 × trig thing

$r^4 + 2r^2 + 1 = 0 \qquad (r^2+1)^2 = 0 \qquad r = \pm i, \pm i$

$$(A_2t^4 + A_1t^3 + A_0t^2)\cos t + (B_2t^4 + B_1t^3 + B_0t^2)\sin t$$

$$y_c = \underbrace{c_1 \cos t + c_2 \sin t}_{\text{from } e^{it}} + \underbrace{c_3 t \cos t + c_4 t \sin t}_{\text{from } te^{it}}$$

Try

$$y_p = t^2\left[\underbrace{(A_2t^2 + A_1t + A_0)\cos t + (B_2t^2 + B_1t + B_0)\sin t}_{A_1 t\cos t,\ A_0\cos t,\ B_1 t\sin t,\ B_0 \sin t \text{ all solve homog!}}\right]$$

$A_0 t \cos t$, $B_0 t \sin t$ solve homog.

*(10 pts.)* Rewrite the third order equation $y^{(3)} + 2y'' + 3y' + 4y = \cos(t)$ as a $(3 \times 3)$ system of first order equations. Suggestion: Let $u_1 = y$, etc.

$u_2 = y'$

$u_3 = y''$

$$\begin{cases} u_1' = y' = u_2 \\ u_2' = y'' = u_3 \\ u_3' = y''' = -2\underset{\underset{y''}{\uparrow}}{u_3} - 3\underset{\underset{y'}{\uparrow}}{u_2} - 4\underset{\underset{y}{\uparrow}}{u_1} + \cos t \end{cases}$$

(10 *pts.*) Find the general solution to

$$y'' + 2y' + y = \underline{\underline{e^{-x}}} + \underline{\underline{e^{x}}}.$$

$r^2 + 2r + 1 = 0$

$r = -1, -1$

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_p = x^2\left[Ae^{-x}\right] + Be^{x}$

$y_p' = 2xAe^{-x} - x^2Ae^{-x} + Be^{x}$

$y_p'' = 2Ae^{-x} + 2(2x)(-e^{-x})A + x^2Ae^{-x}$

$= 2Ae^{-x} - 4xAe^{-x} + x^2Ae^{-x}$

$A = \frac{1}{2}, \quad B = \frac{1}{4}$

*(10 pts.)* The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$r = -2, -2$

$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

is defective. It has only one eigenvalue $r = 2$, which has a one dimensional eigenspace spanned by the eigenvector $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, one solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x_1} = \mathbf{a}e^{2t}$. Find a second linearly independent solution to the system.

$\vec{x}_2 = \vec{a} t e^{rt} + \vec{b} e^{rt}$ $\qquad r = 2, \quad \vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\boxed{(\mathbb{A} - \underset{r=2}{r}\mathbb{I})\vec{b} = \vec{a}}$$

$$\left[\begin{array}{cc|c} 1-2 & -1 & 1 \\ 1 & 3-2 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$b_2$ free

$\vec{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ works great

$b_1 + b_2 = -1$

$b_1 = -1 - b_2$

Let $b_2 = c$.

$\vec{b} = \begin{pmatrix} -1-c \\ c \end{pmatrix}$