Lecture 30 5.7 Nonhomog systems, Undet Coeff, Var of Par

HW 29 due in MyLab

Nonhomogeneous linear systems: $\vec{x}' = \mathbb{P}(t)\vec{x} + \underline{\vec{g}(t)}$

EX: $\begin{cases} \dfrac{dx_1}{dt} = p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ \dfrac{dx_2}{dt} = p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{cases}$

Wronskian $W(t) = Ce^{\int p_{11}(t) + p_{22}(t)\,dt}$ Abel's formula

Homog system: $\vec{x}' = \mathbb{P}(t)\vec{x}$ $(*)$

Step 1: Solve homog syst. (2x2): get $\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2$

Step 2: Get one particular sol'n: $\vec{x}_p$ such that

$$\vec{x}_p' = \mathbb{P}(t)\vec{x}_p + \vec{g}(t)$$

Fact: Gen'l sol'n to $\vec{x}' = \mathbb{P}\vec{x} + \vec{g}$ $(+)$ is

$$\vec{x} = (c_1\vec{x}_1 + c_2\vec{x}_2) + \vec{x}_p$$

Why: Suppose $\vec{x}$ solves $(+)$

$$- \begin{cases} \vec{x}' = \mathbb{P}\vec{x} + \vec{g} \\ \vec{x}_p' = \mathbb{P}\vec{x}_p + \vec{g} \end{cases}$$

$(\vec{x} - \vec{x}_p)' = \mathbb{P}(\vec{x} - \vec{x}_p) + \vec{0}$ Aha! $\vec{x} - \vec{x}_p$ solves $\underbrace{(*)}_{\text{homog}}$

So $\vec{x} - \vec{x}_p = c_1\vec{x}_1 + c_2\vec{x}_2$ for some c's.

One more thing to show: Things of form $c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p$ all solve (+). Easy.

<u>EX:</u> $$\vec{x}\,' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{x} + \begin{pmatrix}1\\2\end{pmatrix}e^{2t}$$

<u>Step 1</u>: Homog sol^n: $\vec{x}\,' = \mathbb{A}\vec{x}$ is

$$\vec{x}_c = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}$$

<u>Step 2</u>: 2 <u>not</u> an e.val, so the Method of Undet Coeff is the way to go.

Try $\boxed{\vec{x}_p = \vec{b}\,e^{2t}}$ and force it!

$$\vec{x}_p\,' = 2\vec{b}e^{2t} \overset{\text{want}}{=} \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{b}\,e^{2t} + \begin{pmatrix}1\\2\end{pmatrix}e^{2t}$$

Divide out e^{2t}: $$2\vec{b} = \mathbb{A}\vec{b} + \begin{pmatrix}1\\2\end{pmatrix}$$

$$-\begin{pmatrix}1\\2\end{pmatrix} = \underbrace{\underbrace{\mathbb{A}\vec{b} - 2\vec{b}}_{(\mathbb{A}-2\mathbb{I})\vec{b}}}_{\text{Aha! det} \neq 0 \text{ because 2 not e.val}}$$

Cramer's rule spits out a $\vec{b}$.

$$(\mathbb{A} - 2\mathbb{I})\vec{b} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\left[\begin{array}{cc|c} -3-2 & 1 & -1 \\ 1 & -3-2 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -5 & 1 & -1 \\ 1 & -5 & -2 \end{array}\right] \qquad \text{Cramer's}: \quad b_1 = \frac{7}{24}, \quad b_2 = \frac{11}{24}$$

$$\vec{x}_p = \frac{1}{24}\begin{pmatrix}7\\11\end{pmatrix}e^{2t}$$

<u>Step 3</u>: Gen^{l} Sol^{n}: $c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \frac{1}{24}\begin{pmatrix}7\\11\end{pmatrix}e^{2t}$

<u>Method of Undet coeff</u>: $\vec{x}' = \mathbb{A}\vec{x} + \vec{g}$

$\mathbb{A}$: const coeff

$\vec{g}$: special form

$\vec{g}$	Try $\vec{x}_p$	
$\begin{pmatrix}3\\4\end{pmatrix}e^{rt}$	$\vec{b}e^{rt}$	$\leftarrow$ danger if r is an e.val
$\begin{pmatrix}3\\4\end{pmatrix}t^2$	$\vec{b}_2t^2 + \vec{b}_1t + \vec{b}_0$	$\leftarrow$ danger if a constant vector $r=0$ is an e.val
$\begin{pmatrix}3\\4\end{pmatrix}\cos\omega t$ or $\begin{pmatrix}3\\4\end{pmatrix}\sin\omega t$	$\vec{a}\cos\omega t + \vec{b}\sin\omega t$	$\leftarrow$ danger $r = \pm\omega i$ e.val
$\begin{pmatrix}3\\4\end{pmatrix}t^5e^{2t}\sin 3t$	Guess it!	

<u>When in danger, use the Method of Variation of Parameters</u>

<u>Step 1</u>: Get homog sol^{n} (even if $\mathbb{P}(t)$ is not const coeff)

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

<u>Step 2</u>: Look for part. sol^{n} $\vec{x}_p = \underbrace{u_1\vec{x}_1 + u_2\vec{x}_2}_{}$

u_1, u_2: fcns of t to be determined

$$\vec{x}_p{}' = u_1'\vec{x}_1 + \underline{\underline{u_1\vec{x}_1{}'}} + \underline{\underline{u_2'\vec{x}_2}} + u_2\vec{x}_2 \overset{\text{want}}{=} \mathbb{P}(u_1\vec{x}_1 + u_2\vec{x}_2) + \vec{g}$$

$$= \underline{\underline{u_1\mathbb{P}\vec{x}_1}} + \underline{\underline{u_2\mathbb{P}\vec{x}_2}} + \vec{g}$$

$$\boxed{u_1'\vec{x}_1 + u_2'\vec{x}_2 = \vec{g}}$$

$$\left[\vec{x}_1 \; , \; \vec{x}_2\right]\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \vec{g}$$

$$\boxed{\mathbb{X}\,\vec{u}' = \vec{g}}$$

$\mathbb{X}$ Fund. matrix for homog

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\boxed{\vec{u}' = \mathbb{X}^{-1}\vec{g}}$$

EX: $\vec{x}' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ ← Hmmm. Undet. Coeff

$\vec{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}e^t$

Try $\vec{x}_p = \vec{a} + \vec{b}e^t$?

Step 1: Homog sol^n $\vec{x}' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\vec{x}$ is

$$\vec{x} = c_1\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t}_{\vec{x}_1} + c_2\underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}}_{\vec{x}_2}$$

Danger! $r=1$ is an e.val!

Step 2: Use Var of Par.

$\vec{g} = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$

$$\mathbb{X} = \left[\vec{x}_1 , \vec{x}_2\right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t , \begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}\right]$$

$$= \begin{bmatrix} e^{t} & 3e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix}$$

$$\mathbb{X}^{-1} = \frac{1}{\underbrace{\left(e^{t}\cdot 2e^{2t} - 3e^{2t}\cdot e^{t}\right)}_{-e^{3t}}} \begin{bmatrix} 2e^{2t} & -3e^{2t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$\mathbb{X}^{-1} = \begin{bmatrix} -2e^{-t} & 3e^{-t} \\ e^{-2t} & -e^{-2t} \end{bmatrix}$$

$$\vec{u}' = \mathbb{X}^{-1}\vec{g} = \begin{bmatrix} -2e^{-t} & 3e^{-t} \\ e^{-2t} & -e^{-2t} \end{bmatrix} \begin{pmatrix} 1 \\ e^{t} \end{pmatrix}$$

$$\vec{u}' = \begin{pmatrix} -2e^{-t}+3 \\ e^{-2t}-e^{-t} \end{pmatrix}$$

no +c's

$$\begin{cases} u_1' = -2e^{-t}+3 & u_1 = \int -2e^{-t} + 3\,dt = 2e^{-t}+3t \\ u_2' = e^{-2t}-e^{-t} & u_2 = \int e^{-2t} - e^{-t}\,dt = -\frac{1}{2}e^{-2t}+e^{-t} \end{cases}$$

Finally $\vec{x}_p = u_1\vec{x}_1 + u_2\vec{x}_2 = \left(2e^{-t}+3t\right)\begin{pmatrix}1\\1\end{pmatrix}e^{t} + \left(-\frac{1}{2}e^{-2t}+e^{-t}\right)\begin{pmatrix}3\\2\end{pmatrix}e^{2t}$

$$= \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$$

Formula to memorize: $\vec{u}' = \mathbb{X}^{-1}\vec{g}$

<u>Remark:</u> For $(n \times n)$ problems $(n > 3)$

might be much less work to solve system

$$\mathbb{X}\vec{u}' = \vec{g}$$

by the method of elimination.

<u>Fun fact:</u> $y'' + p(t)y' + q(t)y = R(t)$

Turn into a 2x2 system: $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

$$\begin{cases} x_1' = y' = x_2 \\ \vec{x}_2 = y'' \underset{\substack{\uparrow \\ \text{from ODE}}}{=} -p\,x_2 - q\,x_1 + R \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\vec{x} + \begin{pmatrix} 0 \\ R \end{pmatrix}$$

Get eqns $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \leftarrow \text{\underline{Chose} this before!} \\ u_1' y_1' + u_2' y_2' = R \end{cases}$