Lecture 31 7.1 Laplace transform

HW30 due in MyLab
HW29W due in GS

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

f function of t for $t \geq 0$

F a new function of s for $s >$ (something)

[$\mathbb{X}$ Fundamental matrix
$\vec{x}_p = \mathbb{X}\vec{u}$ where $\vec{u}' = \mathbb{X}^{-1}\vec{g}$

Book likes $\mathbb{X} = e^{At} = \Phi \leftarrow$ normalized fund matrix: $\Phi(0) = \mathbb{I}$

Huge facts: 1) $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Note: $\mathcal{L}[f] = F$

So $\mathcal{L}[f''(t)] = \mathcal{L}[(f')'] = s\underbrace{\mathcal{L}[f']}_{sF(s)-f(0)} - f'(0)$

$$= s^2F(s) - sf(0) - f'(0)$$

2) $\mathcal{L}$ is a <u>linear</u> operator: $\mathcal{L}[c_1f_1 + c_2f_2] = c_1\mathcal{L}[f_1] + c_2\mathcal{L}[f_2]$

Easy: $\int$ is a linear op.

3) If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$ (for some M), then f_1 and f_2 must be the <u>same</u>!

Consequently, $\mathcal{L}^{-1}$ makes sense.

Master plan: $Ly'' + Ry' + \frac{1}{C}y = e(t)$ $\begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$

Hit eqn with $\mathcal{L}$:

$$L\cdot\left[s^2Y(s) - s\underbrace{y(0)}_{y_0} - \underbrace{y'(0)}_{y_0'}\right] + R\left[sY(s) - \underbrace{y(0)}_{y_0}\right] + \frac{1}{C}Y(s) = E(s)$$

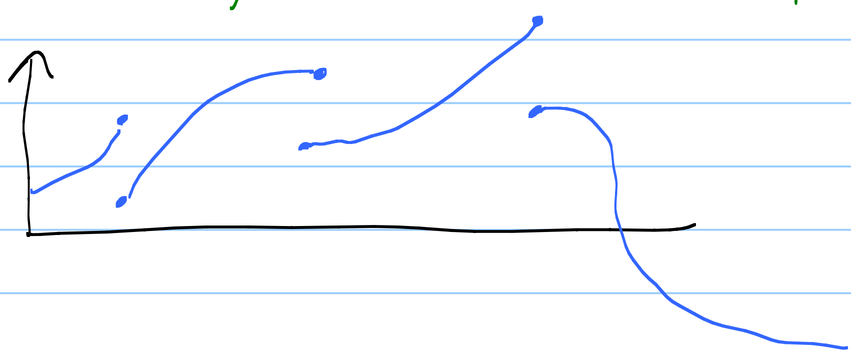
an algebra eqn! Solve for $Y(s)$.

Get sol^n to ODE: $y(t) = \mathcal{L}^{-1}[Y(s)]$.

Exciting thing: $\mathcal{L}$: ODE's $\longrightarrow$ algebra.

↑ solve, then undo $\mathcal{L}$.

Requirements: 1) Need $f(t)$ to be piecewise cont.



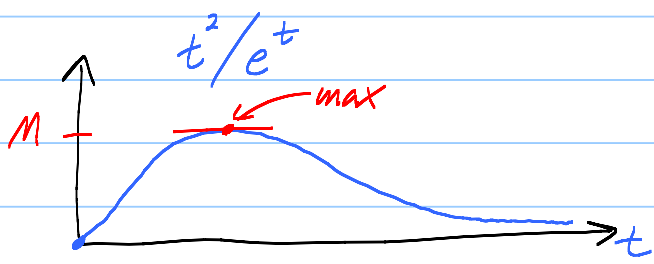
2) Need f to be of "exponential type", meaning there are constants $M>0$ and $K>0$ such that

$$|f(t)| \le M e^{Kt} \text{ for } t \ge 0.$$

EX: $t^2 e^{3t} \cos 2t$

Hmmm. $\frac{t^2}{e^t} \longrightarrow 0$ as $t \to \infty$.

$|\cos 2t| \le 1$ all t.



$\frac{t^2}{e^t} \le M$ for $t > 0$.

$t^2 \le M e^t$

$$\left| t^2 e^{3t} \cos 2t \right| \le (Me^t) e^{3t} \cdot 1 \le M e^{4t} \text{ for } t > 0.$$

$K = 4$.

Consequently, $\mathcal{L}[t^2 e^{3t} \cos 2t]$ makes sense for $s > 4$.

EX: $e^{(e^t)}$ is not of exp. type.

Start our table: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s>a$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \lim_{B\to\infty} \int_0^B = \lim_{B\to\infty} \left[\frac{1}{(a-s)} e^{(a-s)t} \right]_0^B$$

$$= \lim_{B\to\infty} \left[\frac{1}{(a-s)} e^{(a-s)B} - \frac{1}{(a-s)} e^0 \right]$$

need $(a-s)<0$ for this to not blow up.

Aha! When $s>a$, it goes $\to 0$,

$$= \frac{1}{s-a} \text{ when } s>a.$$

$$\mathcal{L}[1] = \int_0^\infty 1 \cdot e^{-st}\,dt = \lim_{B\to\infty} \left[\frac{1}{(-s)} e^{-sB} - \frac{1}{(-s)} e^0 \right]$$

$\to 0$ as $B\to\infty$ if $s>0$.

$$= \frac{1}{s} \text{ when } s>0.$$

$$\mathcal{L}[t^2] = \int_0^\infty \underbrace{t^2}_{u} \underbrace{e^{-st}\,dt}_{dv}$$

$u = t^2 \quad du = 2t\,dt$

$v = \int e^{-st}\,dt = -\frac{1}{s} e^{-st}$

$$= uv - \int v\,du$$

$$= \lim_{B\to\infty} \left[uv \Big|_0^B - \int_0^B v\,du \right]$$

$$= \lim_{B\to\infty} \left[(t^2)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^B - \int_0^B \left(-\frac{1}{s}e^{-st}\right)(2t\,dt) \right]$$

ouch! Int. by parts again!

$u = t \quad v = e^{-st}$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \quad \text{for} \quad s > 0.$$

$$\mathcal{L}[t] = \frac{1}{s^2}, \quad \mathcal{L}[t^2] = \frac{2}{s^3}$$

EX: $\mathcal{L}[\cosh bt] = \mathcal{L}\left[\frac{1}{2}(e^{bt} + e^{-bt})\right]$

$$= \frac{1}{2}\mathcal{L}[e^{bt}] + \frac{1}{2}\mathcal{L}[e^{-bt}]$$

$$= \frac{1}{2}\frac{1}{s-b} + \frac{1}{2}\frac{1}{s-(-b)}$$

$s > b$ $\quad$ $s > -b$

Need $s > b$ for both to make sense

$$= \frac{1}{2}\frac{(s+b)+(s-b)}{(s-b)(s+b)} = \frac{s}{s^2-b^2}, \quad s > b$$

$$\mathcal{L}[\sinh bt] = \mathcal{L}\left[\frac{1}{2}(e^{bt} - e^{-bt})\right] = \frac{b}{s^2-b^2}, \quad s > b.$$

$$\mathcal{L}[\sin bt] = \int_0^\infty (\sin bt)\, e^{-st}\, dt$$ ← integrate by parts twice, solve for $\int$

Hmmm. Integral solves $\frac{du}{dt} = e^{-st}\sin bt$

Briliant idea! Use Meth of Undet Coeff.

Try $u = Ae^{-st}\cos bt + Be^{-st}\sin bt$

Then $\frac{du}{dt} = -sAe^{-st}\cos bt - bAe^{-st}\sin bt$
$-sBe^{-st}\sin bt + bBe^{-st}\cos bt$

$= \underbrace{(-sA + bB)}_{=0} e^{-st}\cos bt + \underbrace{(-sB - bA)}_{1} e^{-st}\sin bt$

wait ↓

$= e^{-st}\sin bt$

$$\begin{cases} -sA + bB = 0 \\ -bA - sB = 1 \end{cases}$$

Cramer's :

$$A = \frac{\det\begin{bmatrix} 0 & b \\ 1 & -s \end{bmatrix}}{\det\begin{bmatrix} -s & b \\ -b & -s \end{bmatrix}} = \frac{-b}{s^2+b^2}$$

$$B = \frac{\det\begin{bmatrix} -s & 0 \\ -b & 1 \end{bmatrix}}{\det\begin{bmatrix} -s & b \\ -b & -s \end{bmatrix}} = \frac{-s}{s^2+b^2}$$

Cool thing!

$$\mathcal{L}[\sin bt] = \mathcal{L}\left[\tfrac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)\right]$$

$$= \frac{1}{2i}\cdot\frac{1}{s-bi} - \frac{1}{2i}\cdot\frac{1}{s-(-bi)}$$

$$= \frac{b}{s^2+b^2} \quad \text{for} \quad s > b.$$

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}, \qquad s > b$$

$$\mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}, \qquad s > b$$