Lecture 32 7.2 Solving ODEs with Laplace Transf.

HW 31 due in MyLab
No class on Monday

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\sin bt$	$\frac{b}{s^2+b^2}$	$(s>0)$
$\cos bt$	$\frac{s}{s^2+b^2}$	$(s>0)$
1	$\frac{1}{s}$	$(s>0)$
t	$\frac{1}{s^2}$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$	
$\mathcal{L}[f'(t)]$	$sF(s)-f(0)$	
$\mathcal{L}[f''(t)]$	$s^2F(s)-sf(0)-f'(0)$	

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[t^n] = \int_0^\infty \underbrace{t^n}_{u}\,\underbrace{e^{-st}\,dt}_{dv}$$

$u = t^n \qquad du = nt^{n-1}\,dt$

$dv = e^{-st}\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \underbrace{t^n\left(-\tfrac{1}{s}e^{-st}\right)\Big|_0^\infty}_{0} - \int_0^\infty \left(-\tfrac{1}{s}e^{-st}\right) n t^{n-1}\,dt$$

$$= \frac{n}{s}\underbrace{\int_0^\infty t^{n-1} e^{-st}\,dt}_{\mathcal{L}[t^{n-1}]}$$

Reduction formula: $\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$

$$= \frac{n(n-1)}{s\cdot s}\mathcal{L}[t^{n-2}] = \cdots$$

$$= \frac{n(n-1)(n-2)\cdots 1}{s^n}\mathcal{L}[1]$$

$$= \frac{n!}{s^{n+1}} \checkmark$$

EX: $y'' + 5y' + 6y = 0$ $\begin{cases} y(0)=3 \\ y'(0)=5 \end{cases}$

Hit with $\mathcal{L}$:

$$\mathcal{L}[y''] + 5\mathcal{L}[y'] + 6\mathcal{L}[y] = 0$$

$$\left(s^2Y(s) - s\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{5}\right) + 5\left(sY - \underbrace{y(0)}_{3}\right) + 6Y = 0$$

$$(s^2+5s+6)Y = 3s+20$$

$$Y = \frac{3s+20}{s^2+5s+6} = \frac{3s+20}{(s+2)(s+3)}$$

Need $\mathcal{L}^{-1}$ of this!

$$\overset{\text{Part. Frac.}}{=} \frac{A}{s+2} + \frac{B}{s+3}$$

$\frac{1}{s-a}$ in table!

Partial fractions: 1) Mult. by denom:

$$3s+20 = A(s+3) + B(s+2)$$

2) Collect coeff of powers of s:

$$3s+20 = \underbrace{(A+B)}_{3}s + \underbrace{(3A+2B)}_{20}$$

3) Equate coeff:

4) Solve syst:

$$\begin{cases} A+B=3 \\ 3A+2B=20 \end{cases}$$

$$A = \frac{\det\begin{bmatrix} 3 & 1 \\ 20 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{(-14)}{(-1)} = 14$$

$$B = \frac{\det\begin{bmatrix} 1 & 3 \\ 3 & 20 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{11}{(-1)} = -11$$

So $$Y(s) = \frac{14}{s+2} - \frac{11}{s+3}$$

$$= 14 \cdot \frac{1}{s-(-2)} - 11\,\frac{1}{s-(-3)}$$

$$= \mathcal{L}\left[\underbrace{14e^{-2t} - 11e^{-3t}}\right]$$

Sol'n $y(t) = \mathcal{L}^{-1}[Y(s)]$

Dealing with $\dfrac{ds+e}{as^2+bs+c} = Y(s)$

1) $as^2+bs+c = a(s-r_1)(s-r_2) \leftarrow r_1 \neq r_2$ real

$$Y(s) = \frac{A}{s-r_1} + \frac{B}{s-r_2}$$

2) $as^2+bs+c = a(s-r)^2 \leftarrow r$ real

$$Y(s) = \frac{A}{\underbrace{(s-r)^2}} + \frac{B}{\underbrace{s-r}_{\mathcal{L}[Be^{rt}]}}$$

partial frac

Hmmm. $\stackrel{?}{=} \mathcal{L}[Ate^{rt}]$ (yes!)

3. as^2+bs+c has complex roots.

Step 1: Complete the square in denom:

Completing the square: $as^2+bs+c = a\left(s^2 + \underbrace{\tfrac{b}{a}}_{B}s + \underbrace{\tfrac{c}{a}}_{C}\right)$

$$s^2 + Bs + C = \left(s + \frac{B}{2}\right)^2 + \left(C - \frac{B^2}{4}\right)$$

$$s^2 + \underbrace{2\_\_}_{\text{want } = B}s + \_\_^2$$

EX: $s^2 + 4s + 13 = \left(s + \frac{4}{2}\right)^2 + \underline{\quad 9 \quad}$

$$= s^2 + 4s + 4 \quad + \underline{\quad 9 \quad}$$

$$= (s+2)^2 + 9 \leftarrow 3^2$$

Hmmm:

f	F
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\dfrac{(s-a)}{(s-a)^2+b^2}$

EX: $\dfrac{3s+4}{s^2+4s+13} = \dfrac{3s+4}{(s+2)^2+3^2}$ $\qquad a=2$, $b=3$

Hmmm. Wish I had $(s+2)$ in num. instead of s.

Give it to yourself like this:

$$s = [(s+2)-2]$$

Now $\dfrac{3s+4}{s^2+4s+13} = \dfrac{3[(s+2)-2]+4}{(s+2)^2+3^2}$

give yousel f a 3!

$$= 3 \cdot \frac{(s+2)}{(s+2)^2+3^2} - \frac{2}{3} \frac{3}{(s+2)^2+3^2}$$

$$= \mathcal{L}\left[3e^{-2t}\cos 3t - \tfrac{2}{3}e^{-2t}\sin 3t\right]$$

Why $\mathcal{L}[f'(t)] = sF(s) - f(0)$:

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \underbrace{e^{-st}}_{u} \ \underbrace{f'(t)\, dt}_{dv}$$

$du = -se^{-st}$ $\qquad$ $v = \int f'\, dt = f$

$$= uv\Big|_0^\infty - \int_0^\infty v\, du$$

$$= \underbrace{e^{-st} f(t)\Big|_0^\infty}_{|f(t)| \le Me^{kt}} - \int_0^\infty f(t)\left[-se^{-st}\, dt\right]$$

$\longrightarrow 0$ as $t \to \infty$

if $s > M$

$$= 0 - 1 \cdot f(0) + s \underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}_{F(s)} \ \checkmark$$

What about $\mathcal{L}[\underbrace{\text{anti-derivative of } f}_{g(t) = \int_0^t f(\tau)\, d\tau}]$?

$g(0) = \int_0^0 f(\tau)\, d\tau = 0$

Hmmm. $g'(t) = f(t)$ Fund. Thm. Calc.

$\mathcal{L}[g'(t)] = \mathcal{L}[f(t)]$

$$= s\mathcal{L}[g] - \underbrace{g(0)}_{0} = F(s)$$

$$\mathcal{L}[\underbrace{g(t)}_{\int_0^t f(\tau)\,d\tau}] = \frac{F(s)}{s}$$

f	F
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\mathcal{L}[f^{(n)}(t)]$	$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$