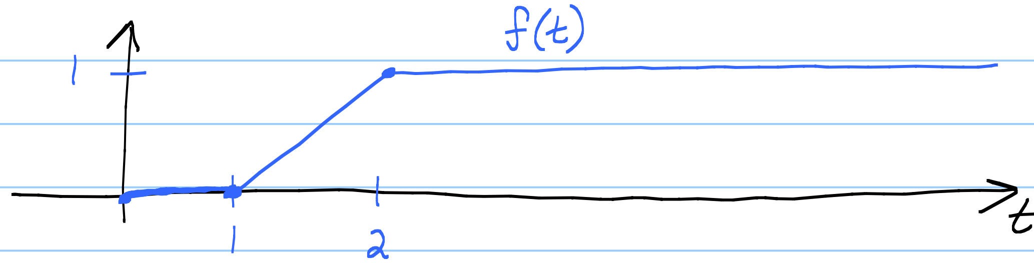
Lecture 33 on 7.3 s-shift, t-shift rules

HW32 due in MyLab

Piecewise defined fcns

EX:

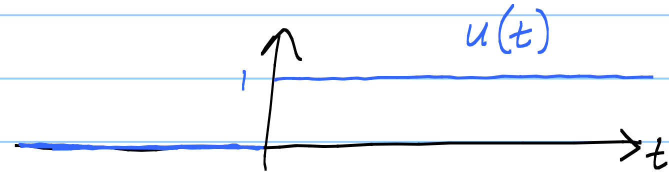


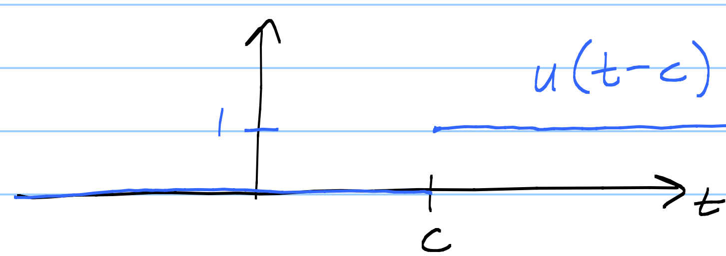
$$f(t) = \begin{cases} 0 & t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1 & t \ge 2 \end{cases}$$

Today: $\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} \, dt$

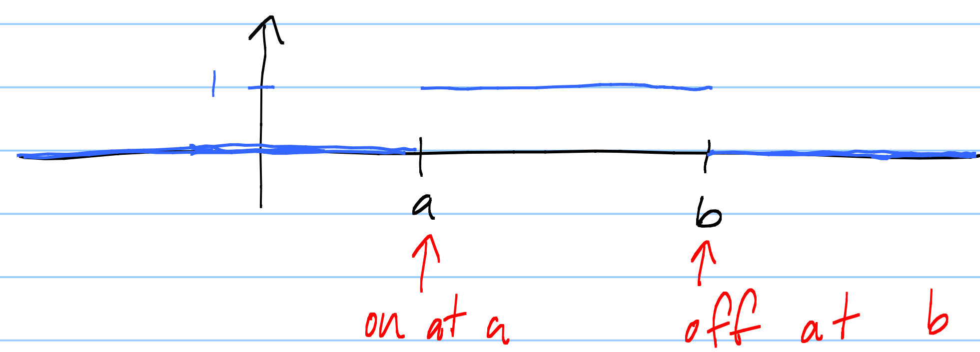
$$= 0 + \int_1^2 (t-1) e^{-st} \, dt + \int_2^\infty 1 \cdot e^{-st} \, dt$$

Def^n: The "Heaviside" fuction, or unit step function $u(t)$ or $u_0(t)$

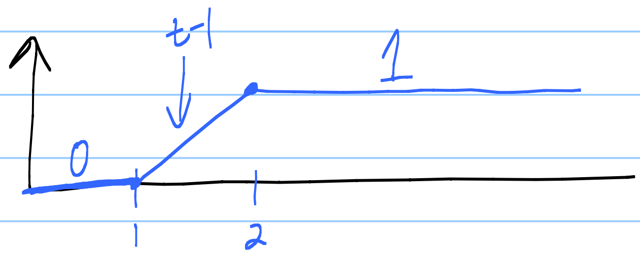




$\overset{\text{def}^n}{=} u_c(t)$



$u_a(t) - u_b(t)$



$f(t)$

$$f(t) = 0\cdot u_0(t) + \underbrace{(t-1)\left[u_1(t) - u_2(t)\right]}_{\text{Turn on } (t-1) \text{ at } t=1 \text{ and off at } t=2} + 1\cdot u_2(t)$$

$$f(t) = (t-1)u_1(t) + \left[-(t-1)+1\right]u_2(t) = (t-1)u_1(t) - (t-2)u_2(t)$$

Two important rules: 1) s-shifting rule: Suppose $\mathcal{L}[f(t)] = F(s)$.

Then $\mathcal{L}\left[e^{at}f(t)\right] = \underbrace{F(s-a)}_{F \text{ "shifted" by } a}$

2) t-shifting rule:

$$\mathcal{L}\left[\underbrace{u_c(t)}_{\uparrow}\ \underbrace{f(t-c)}_{f \text{ "shifted" by } c}\right] = e^{-cs}F(s)$$

cuts off
f before t=0

Remark: Take $f(t) \equiv 1$ in #2: $\mathcal{L}[1] = \frac{1}{s}$ $\leftarrow f(t) \equiv 1$

So $\mathcal{L}[u_c(t)] = \mathcal{L}[u_c(t)f(t-c)]$

$$= e^{-cs}\cdot\frac{1}{s}$$

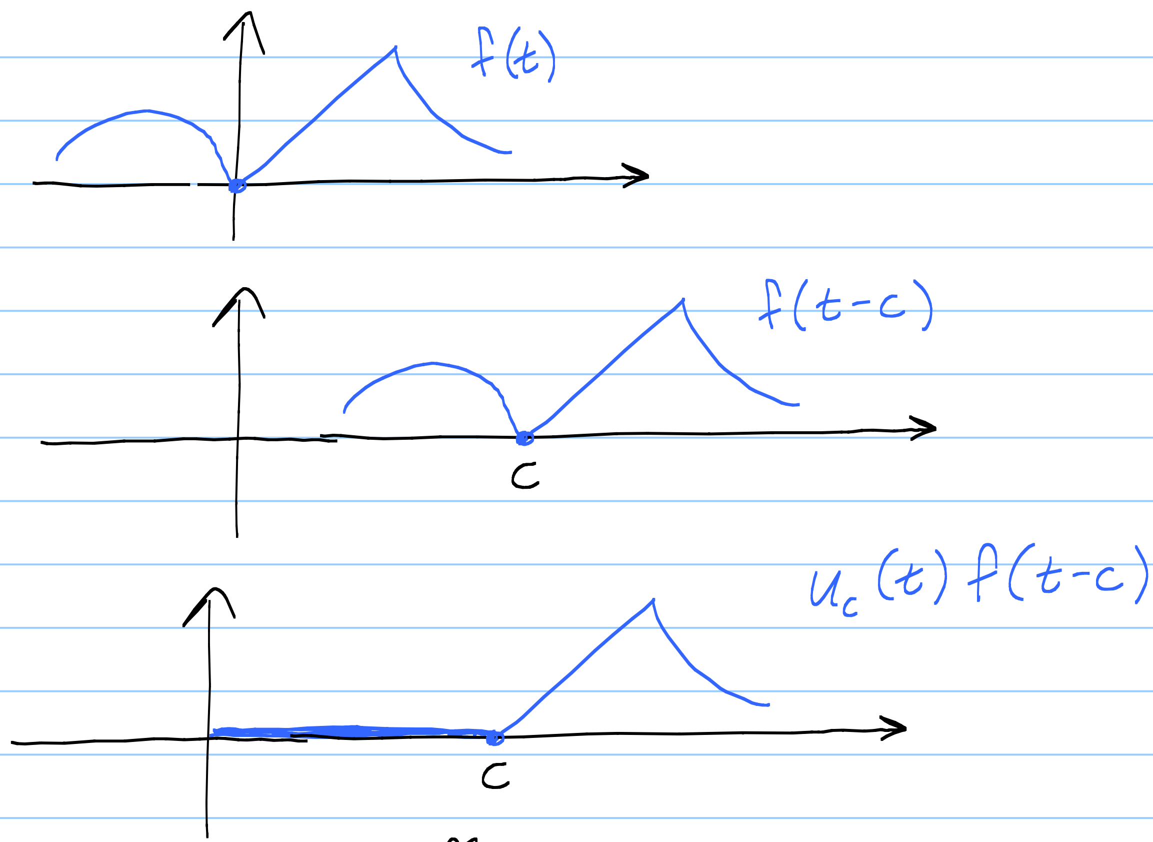
Table: $\boxed{\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}}$

Why 1: $\mathcal{L}\left[e^{at}f(t)\right] = \int_0^\infty e^{at}f(t)e^{-st}\,dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$= F(s-a)$$

Why 2:



$$\mathcal{L}[u_c(t) f(t-c)] = \int_c^\infty f(\underbrace{t-c}_{\tau})\, e^{-st}\, dt$$

$\tau = t - c$

$d\tau = dt$

so

$t = \tau + c$

When $t = c$: $\tau = 0$

As $t \to \infty$, so does τ

$$= \int_{\tau=0}^\infty f(\tau)\, \underbrace{e^{-s(\tau+c)}}_{e^{-s\tau} e^{-cs}}\, d\tau$$

$$= e^{-cs} \underbrace{\int_0^\infty f(\tau)\, e^{-s\tau}\, d\tau}_{F(s)}$$

Important formulas:

$$\mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}$$

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2}$$

s-shift rule:

$$\mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}[e^{at}\cos bt] = \frac{(s-a)}{(s-a)^2+b^2}$$

EX:

$$\frac{3s+4}{\underbrace{s^2+2s+2}} = \frac{3\overbrace{[(s+1)-1]}^{s}+4}{(s+1)^2+1}$$

complex roots

Complete the square

$\frac{1}{2}\cdot(2)$

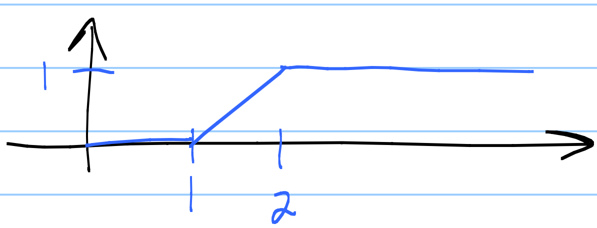
$\boxed{a=-1}$

$b=1$

$$= 3\cdot\frac{(s+1)}{(s+1)^2+1^2} + \frac{1}{(s+1)^2+1^2}$$

$$= \mathcal{L}\left[3e^{-t}\cos t + e^{-t}\sin t\right]$$

EX:



$$f(t) = (t-1)u_1(t) - (t-2)u_2(t)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[u_1(t)\underbrace{g(t-1)}] - \mathcal{L}[u_2(t)\underbrace{h(t-2)}]$$

where $g(t)=t$ where $h(t)=t$

t-shift rule: $$= e^{-1\cdot s}\cdot\frac{1}{s^2} - e^{-2\cdot s}\cdot\frac{1}{s^2}$$

$$= \frac{(e^{-s} - e^{-2s})}{s^2}$$

EX: Trick: $t = (t-c) + c$

$$\mathcal{L}[u_3(t)\, t^2] = \mathcal{L}[u_3(t)\, [(t-3)+3]^2]$$

If only we had $t-3$ here instead!

$$= \mathcal{L}[u_3(t)\, \underbrace{[(t-3)^2 + 6(t-3) + 9]}_{= g(t-3)}]$$

where $g(t) = t^2 + 6t + 9$

$$= e^{-3s}\, G(s)$$

$$= e^{-3s}\left(\frac{2!}{s^3} + 6 \cdot \frac{1}{s^2} + 9 \cdot \frac{1}{s}\right)$$

Partial fractions:

$$\frac{s^{10} + 3s^5 + 1}{s^2 (s-3)^3 \underbrace{(s^2+2s+2)^2}_{\text{complex roots}} \underbrace{(s^2+9)}_{\text{complex roots}}}$$

← degree < 11

← degree 11

$s = (s-0)$

$$= \underbrace{\left(\frac{A}{s^2} + \frac{B}{s}\right)}_{\text{for } s^2 \text{ term}} + \underbrace{\left(\frac{C}{(s-3)^3} + \frac{D}{(s-3)^2} + \frac{E}{(s-3)}\right)}_{\text{for } (s-3)^3 \text{ term}}$$

$$+ \underbrace{\left(\frac{Fs+G}{(s^2+2s+2)^2} + \frac{Hs+I}{(s^2+2s+2)}\right)}_{\text{for } (s^2+2s+2)^2} + \left(\frac{Js+K}{s^2+9}\right)$$

<u>Method:</u> 1) Multiply by denom

2) Collect terms in front of powers of s on both sides

3) equate coeff, get a linear system

4) solve the system (guaranteed!)