Lecture 34 on 7.4 Using Laplace Transform to solve IVPs.

HW 33 due in MyLab
HW30w, 31w, 32w due in GS

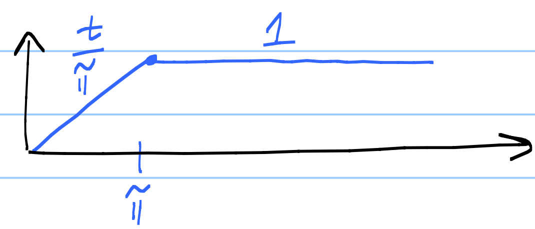
s-shift rule : $\mathcal{L}[e^{at}f(t)] = F(s-a)$

t-shift rule : $\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s)$

$\left(\mathcal{L}[f(t)] = F(s) \right)$

Remark : $f(t)$, $\underbrace{u_0(t)f(t)}_{=u_0(t)f(t-0)}$ are the <u>same</u> from point of view of $\mathcal{L}$ because $\mathcal{L} = \int_0^\infty$

$$\mathcal{L}[u_0(t)f(t-0)] = e^{-0\cdot s}F(s) = F(s) \checkmark$$

Prob: $y'' + y =$ 

$g(t)$

$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

$$g(t) = \underbrace{\frac{t}{\pi}[u_0(t) - u_\pi(t)]}_{\text{Turn on } \frac{t}{\pi} \text{ at time } 0 \text{ and off at time } \pi} + \underbrace{1 \cdot u_\pi(t)}_{\text{Turn on } 1 \text{ at time } \pi}$$

$$g(t) = \frac{t}{\pi}u_0(t) - \underbrace{\left(\frac{t}{\pi} - 1\right)u_\pi(t)}_{-\frac{1}{\pi}(t-\pi)u_\pi(t)}$$

$$-\frac{1}{\pi}\underbrace{(t-\pi)}_{f(t-\pi) \text{ where } f(t)=t}u_\pi(t)$$

$$G(s) = \mathcal{L}[g(t)] = \frac{1}{\pi}\mathcal{L}[t] - \frac{1}{\pi}e^{-\pi s}\underbrace{\mathcal{L}[t]}_{=1/s^2}$$

$$= \frac{1}{\pi}\cdot\frac{1}{s^2} - \frac{1}{\pi}e^{-\pi s}\cdot\frac{1}{s^2}$$

Hit ODE with $\mathcal{L}$:

$$\left[s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right] + Y(s) = G(s)$$

$$(s^2+1)Y(s) = G(s)$$

$$Y(s) = \frac{1}{\pi}\underbrace{\frac{1}{s^2(s^2+1)}}_{H(s)} - \frac{1}{\pi}e^{-\pi s}\underbrace{\frac{1}{s^2(s^2+1)}}_{H(s)}$$

Sol^n:

$$y(t) = \frac{1}{\pi}h(t) - \frac{1}{\pi}\underbrace{u_\pi(t)\,h(t-\pi)}_{\text{via } t\text{-shift rule backwards}}$$

Partial fractions!

$$H(s) = \frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

1) Mult by denom:

$$1 = A(s^2+1) + Bs(s^2+1) + s^2(Cs+D)$$

$$0s^3+0s^2+0s+1 = \underbrace{(B+C)}_{\text{must } =0}s^3 + \underbrace{(A+D)}_{=0}s^2 + \underbrace{(B)}_{=0}s + \underbrace{A}_{1}$$

$A=1$

$B=0$

$A+D=0 \Leftarrow D=-A=-1$

$B+C=0 \Leftarrow C=-B=0$

$$H(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} + \frac{0}{s} + \frac{0s-1}{s^2+1}$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1}$$

So $h(t) = t - \sin t$

Ans:

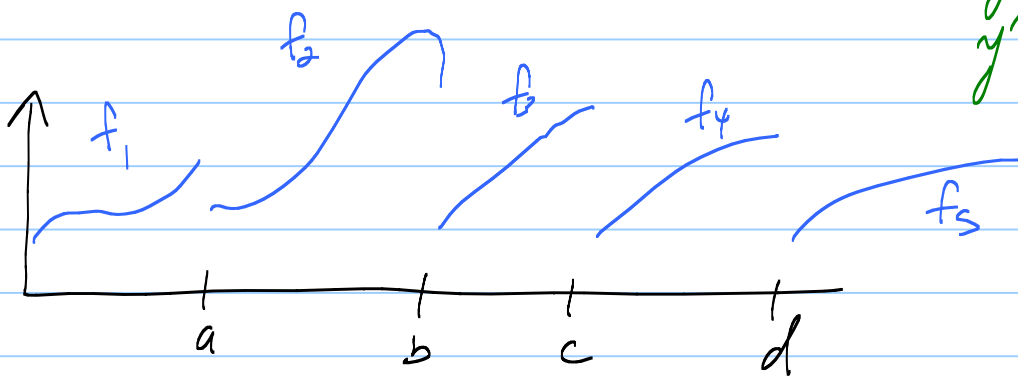
$$y(t) = \frac{1}{\pi}h(t) - \frac{1}{\pi}u_\pi(t)\,h(t-\pi)$$

$$= \frac{1}{\pi}(t-\sin t) - \frac{1}{\pi}u_\pi(t)\left[(t-\pi) - \underbrace{\sin(t-\pi)}_{-\sin t}\right]$$

$$= \begin{cases} \frac{t}{\pi} - \frac{1}{\pi}\sin t & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t \ge \pi \end{cases}$$

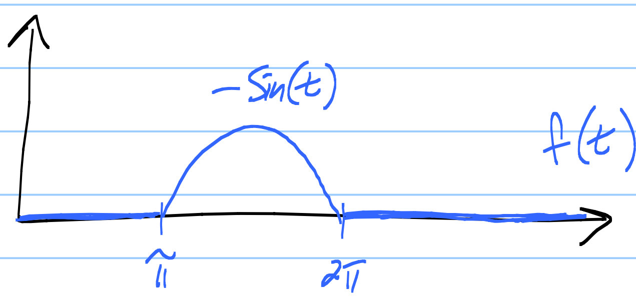
Old fashioned way: Solve IVP on $[0,\pi]$.

Next, solve IVP on $[\pi,\infty)$ using initial cond $\left.\begin{array}{l} y(\pi) \\ y'(\pi) \end{array}\right\}$ from first prob.



$$f_1(t)\left[u_0(t)-u_a(t)\right] + f_2(t)\left[u_a(t)-u_b(t)\right] + f_3(t)\left[u_b(t)-u_c(t)\right] + \cdots$$

EX: (Tricks)



$$(-\sin t)\left[u_{\pi}(t) - u_{2\pi}(t)\right]$$

$$f(t) = u_{\pi}(t)\underbrace{(-\sin t)}_{=\sin(t-\pi)} + u_{2\pi}(t)\underbrace{\sin t}_{=\sin(t-2\pi)}$$

$$F(s) = e^{-\pi s}\cdot\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1}$$

Hmmm: $\sin t = \sin\left((t-\pi)+\pi\right)$

$$= \cos(t-\pi)\underbrace{\sin\pi}_{0} + \sin(t-\pi)\underbrace{\cos\pi}_{-1} = -\sin(t-\pi)$$

✓

EX: $g(t) = u_3(t)e^{2t} \leftarrow$ wish $e^{2(t-3)}$ instead

$$= u_3(t)\, e^{2[(t-3)+3]}$$

$$= u_3(t)\, e^{6}\, \underbrace{e^{2(t-3)}}_{f(t-3) \text{ where } f(t)=e^{2t}}$$

So $G(s) = e^{6}\, e^{-3s} \cdot \frac{1}{s-2}$

EX: $\mathcal{L}[t \sin t] = \cancel{\mathcal{L}[t]\,\mathcal{L}[\sin t]}$ No, no, no!

$$= \int_0^\infty e^{-st}\, t \sin t \, dt \quad \text{ouch!}$$

Hmmm: $f(t) = t \sin t \leftarrow f(0)=0$

$f'(t) = t \cos t + \sin t \leftarrow f'(0) = 0$

$f''(t) = -t \sin t + 2\cos t$

Aha! $\underbrace{\mathcal{L}[f''(t)]} = s^2F(s) - sf(0) - f'(0)$

$$= s^2F(s)$$

$$= \mathcal{L}[-t\sin t + 2\cos t] = -F(s) + 2 \cdot \frac{s}{s^2+1}$$

$=$

Solve for $F(s)$: $F(s) = \dfrac{2s}{(s^2+1)^2}$

Similarly, get $\mathcal{L}[t \cos t]$.

EX: $\mathcal{L}[te^{at}]$

$f(t) = te^{at} \leftarrow f(0) = 0$

$f'(t) = at\,e^{at} + e^{at} \quad (B)$

$$\mathcal{L}[f'(t)] \underset{\substack{\uparrow \\ \text{from (B)}}}{=} a\underbrace{\mathcal{L}[te^{at}]}_{F(s)} + \frac{1}{s-a} \underset{\substack{\uparrow \\ \text{deriv rule}}}{=} sF(s) - f(0)$$

Solve for $F(s)$: $(s-a)F(s) = \dfrac{1}{s-a}$

$$F(s) = \frac{1}{(s-a)^2}$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}$$

How dumb of me! Could have used the s-shift rule directly