Lecture 35 on 7.5 Convolution

HW34 due in MyLab

$$\mathcal{L}[t\sin t] = \cancel{\frac{1}{s^2}\cdot\frac{1}{s^2+1}}$$

No, no, no!

$$\mathcal{L}[t]=\frac{1}{s^2} \qquad \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

True formula: $\mathcal{L}[f*g] = F(s)\,G(s)$

where $$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

= "the convolution of f and g"

$0 \quad \tau \quad t-\tau \quad t$

Discrete version: $$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where $$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Warning: Today's convolution is for Laplace transform.

Convolution for Fourier transform is $c\int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$.

EX: Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$

$$\frac{1}{s^2+1}\cdot\frac{1}{s^2+1} \leftarrow \mathcal{L}[\sin t]$$

Ans: $f*f$ where $f(t)=\sin t$

$$(f*f)(t) = \int_0^t f(\tau)\,f(t-\tau)\,d\tau$$

$$= \int_0^t \sin\tau \, \sin(t-\tau)\, d\tau$$

$$\boxed{\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)}$$

$\alpha = \tau, \ \beta = t-\tau$

$$\sin\tau \, \sin(t-\tau) = \frac{1}{2}\cos(2\tau - t) - \frac{1}{2}\cos t$$

So $$(f*f)(t) = \int_0^t \left(\frac{1}{2}\cos(\underbrace{2\tau - t}_{u}) - \frac{1}{2}\cos t \right) d\tau$$

$u = 2\tau - t$

$du = 2\,d\tau$

When $\tau = 0, \ u = -t$

$\tau = t, \ u = t$

$$= \frac{1}{2}\int_{-t}^{t} \cos u \left(\tfrac{1}{2} du\right) - \left(\frac{1}{2}\cos t\right) \underbrace{\int_0^t 1\, d\tau}_{t}$$

$$= \frac{1}{4}\left[\sin u\right]_{-t}^{t} - \frac{1}{2} t \cos t$$

$$= \frac{1}{4}\left(\sin t - \underbrace{\sin(-t)}_{-\sin t}\right) - \frac{1}{2} t \cos t$$

$$= \frac{1}{2}\sin t - \frac{1}{2}\underbrace{t\cos t}_{\text{resonance!}}$$

$$\mathcal{L}\left[-\frac{1}{2} t \cos t + \frac{1}{2}\sin t\right] = \frac{1}{(s^2+1)^2}$$

See book for $\cos\omega t$, ... formulas.

<u>Remark:</u> $\mathcal{L}[t f(t)] = -F'(s)$ cousin to $\mathcal{L}[f'(t)] = sF(s) - f(0)$

So $\mathcal{L}[t \cos t] = -\dfrac{d}{ds}\left(\dfrac{s}{(s^2+1)}\right)$

Partial Fractions:

1) $\dfrac{A}{s}$ $\quad \mathcal{L}[A]$

2) $\dfrac{B}{s^2}$ $\quad \mathcal{L}[Bt]$

3) $\dfrac{C}{s^3}$ $\quad \mathcal{L}[\frac{C}{2}t^2]$

4) $\dfrac{A}{s-a}$ $\quad \mathcal{L}[Ae^{at}]$

5) $\dfrac{B}{(s-a)^2}$ $\quad \mathcal{L}[Bte^{at}]$

6) $\dfrac{As+B}{(s-a)^2+b^2}$, Combo of $e^{at}\cos bt$, $e^{at}\sin bt$

7) $\dfrac{As+B}{s^2+b^2}$, Combo of $\cos bt$, $\sin bt$

8) $\dfrac{As+B}{(s^2+b^2)^2}$, today's stuff

9) $\dfrac{As+B}{((s-a)^2+b^2)^2}$, today's stuff plus the s-shift

Exciting thing!

$$y'' + y = r(t) \quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$s^2 Y + Y = R(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}[\sin t]} \cdot \underbrace{R(s)}_{\mathcal{L}[r(t)]}$$

So $y(t) = (f * r)(t)$ where $f(t) = \sin t$!

$$= \int_0^t \sin(\tau)\, r(\underbrace{t-\tau}_{\text{hate this here}})\, d\tau$$

Fact: $(f*g) = (g*f)$

why $\mathcal{L}[f*g] = F(s)\,G(s) = G(s)\,F(s) = \mathcal{L}[g*f]$

must be same!

So $y(t) = (r*f)(t) = \boxed{\int_0^t r(\tau) \sin(t-\tau)\, d\tau}$

Fantastic thing: If $r(t)$ is gotten from measurements, can use numerical integration to get $y(t)$!

Why: $\mathcal{L}[f*g] = F(s)\,G(s)$:

$$F(s)\cdot G(s) = \int_0^\infty f(u)\,e^{-su}\,du \int_0^\infty g(v)\,e^{-sv}\,dv$$

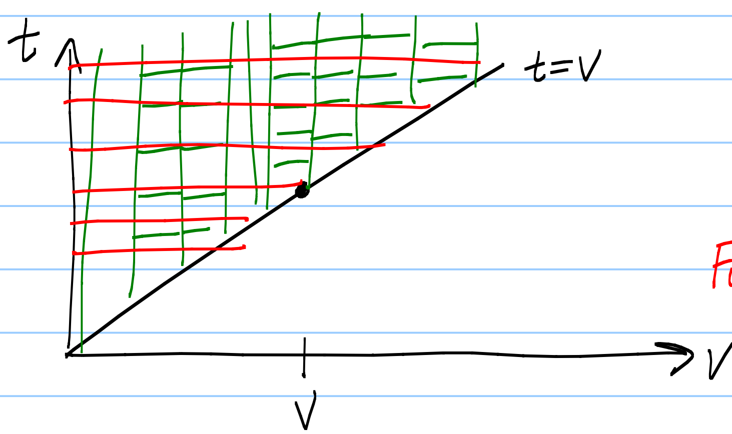
$$\underset{\text{Fubini's}}{=} \int_{v=0}^\infty \left[\int_{u=0}^\infty f(u)\,g(v)\,e^{-s(u+v)}\,dv\right]du$$

$$= \int_{v=0}^\infty g(v)\left[\int_{u=0}^\infty f(u)\,e^{-s\underbrace{(u+v)}_{t}}\,dv\right]du$$

When $u=0,\ t=v$ $\qquad u+v=t \qquad u=t-v$

$u\to\infty,\ t\to\infty$ $\qquad dv=dt$

$$= \int_{v=0}^\infty g(v)\left(\int_{t=v}^\infty f(t-v)\,e^{-st}\,dt\right)du$$



$$\underset{\text{Fubini's}}{=} \int_{t=0}^\infty e^{-st}\underbrace{\int_{v=0}^{t} g(v)\,f(t-v)\,dv}_{\text{Aha! } g*f}\,dt$$

$$= \mathcal{L}[(g*f)] \ !$$

Fun thing: Can solve sytems with $\mathcal{L}.T.$

$$\begin{cases} \dot{x}_1 = 2x_1 - 3x_2 \\ \dot{x}_2 = 5x_1 - x_2 \end{cases}$$

$$\begin{cases} s\underline{X}_1(s) - x_1(0) = 2\underline{X}_1 - 3\underline{X}_2 \\ s\underline{X}_2(s) - x_2(0) = 5\underline{X}_1 - \underline{X}_2 \end{cases}$$

$$\begin{pmatrix} -x_1(0) \\ -x_2(0) \end{pmatrix} = \begin{bmatrix} 2-s & -3 \\ 5 & -1-s \end{bmatrix} \begin{pmatrix} \underline{X}_1(s) \\ \underline{X}_2(s) \end{pmatrix}$$

Use Cramer's rule, etc.

Next time: The Delta fcn. $\leftarrow$ unit impulse fcn.