Lecture 36 on 7.6 The Dirac Delta function

HW 35 due in MyLab

$$F = ma = m\frac{dv}{dt}$$

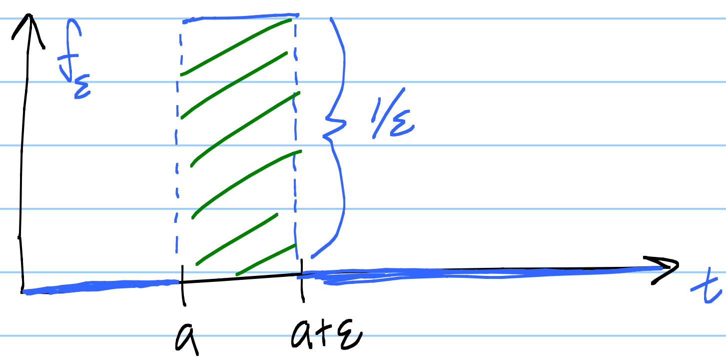
Integrate dx, use $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$$\int F\,dx = m\int v\frac{dv}{dx}\,dx = m\int \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)dx$$
$$= \tfrac{1}{2}\Delta mv^2$$

Work $= \Delta$(Kinetic Energy)

Integrate dt,
$$\underbrace{\int F\,dt}_{\text{impulse}} = m\int\frac{dv}{dt}\,dt = \underbrace{m\Delta v}_{\Delta\text{ Momentum}}$$

Good sledge hammer:



Area $= 1$

Unit impulse

$$f_\varepsilon(t) = \begin{cases} 0 & t \le a,\ t \ge a+\varepsilon \\ \frac{1}{\varepsilon} & a < t < a+\varepsilon \end{cases}$$

Note: $\lim_{\varepsilon \to 0} f_\varepsilon(t) = 0$ for all t.

$$f_\varepsilon(t) = \underbrace{\frac{1}{\varepsilon}\left[u_a(t) - u_{a+\varepsilon}(t)\right]}$$

Turn on $\frac{1}{\varepsilon}$ at time $t=a$
and off at time $t=a+\varepsilon$

$$= \frac{1}{\varepsilon}u_a(t) - \frac{1}{\varepsilon}u_{a+\varepsilon}(t)$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon}\cdot\frac{e^{-as}}{s} - \frac{1}{\varepsilon}\frac{e^{-(a+\varepsilon)s}}{s} = \frac{1}{s}\cdot\underbrace{\frac{e^{-as} - a^{-(a+\varepsilon)s}}{\varepsilon}}_{-\frac{h(a+\varepsilon) - h(a)}{\varepsilon}}$$

where $h(x) = e^{-xs}$

$DQ \longrightarrow -h'(x)\Big|_{x=a} = s e^{-xs}\Big|_{x=a}$

Hmmm. Even though $f_\varepsilon(t) \to 0$ for all t, $= s e^{-as}$

$$F_\varepsilon(s) \longrightarrow e^{-as}$$

Laplace Transform of the ideal unit sledge hammer.

Think: (feel) $\delta_a(t)$ "=" $\lim_{\varepsilon \to 0} f_\varepsilon(t)$

$$\boxed{\mathcal{L}[\delta_a(t)] = e^{-as}}$$

Problem: $y'' + y = f_\varepsilon(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$ | $my'' + cy' + ky = f$

$$s^2 Y + Y = \underbrace{\frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)}_{F_\varepsilon(s)}$$

$$Y = \frac{1}{s^2+1} F_\varepsilon(s) \longrightarrow \frac{1}{s^2+1} e^{-as}$$

In limit as $\varepsilon \to 0$, the y fcns $\longrightarrow$

$$y = \mathcal{L}^{-1}\left[\frac{e^{-as}}{s^2+1}\right]$$

$$y = u_a(t) \sin(t-a)$$

Good sledge hammer: Momentum grows from 0 to 1 between a and $a+\varepsilon$.

"Ideal" sledge hammer: Momentum jumps from 0 to 1 at $t=a$.

Key properties of $\delta_a(t)$: 1) $\delta_a(t)=0$ if $t\neq a$

2) $\int \delta_a(t)\,dt = 1$

3) If h is continuous, then

$$\underbrace{\int h(t)\,\delta_a(t)\,dt = h(a)}$$

means "in the sense of distributions"

$$\lim_{\varepsilon\to 0}\int h(t)\,f_\varepsilon(t)\,dt = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^{a+\varepsilon} h(t)\,dt = h(a)$$

Consequence of (3): $\displaystyle\int_0^\infty e^{-st}\,\delta_a(t)\,dt = h(a) = e^{-as}$

where $h(t)=e^{-ts}$

EX: $y''+4y'+13y=\delta_\pi(t)$ $\quad\begin{cases} y(0)=0 \\ y'(0)=0\end{cases}$

$$s^2Y+4sY'+13Y=e^{-\pi s}$$

$$Y=\frac{e^{-\pi s}}{s^2+4s+13}$$

$$=e^{-\pi s}\cdot\underbrace{\frac{1}{s^2+4s+13}}_{H(s)}$$

s-shift rule

$$\mathcal{L}[e^{at}f(t)]=F(s-a)$$

t-shift rule

$$\mathcal{L}[u_c(t)f(t-c)]=e^{-cs}F(s)$$

Ans: $y(t) = u_{\pi}(t)\, h(t-\pi)$

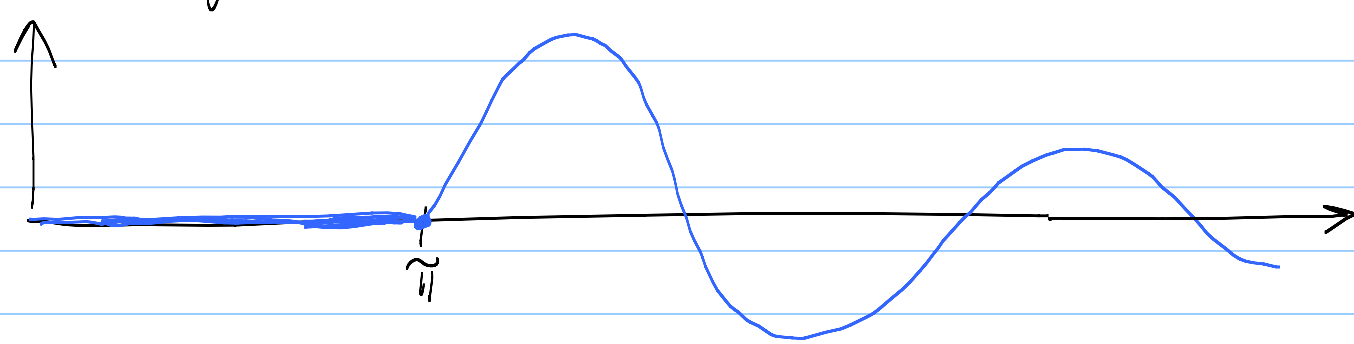
Find h: $\dfrac{1}{s^2+4s+13} = \dfrac{1}{(s+2)^2+9} = \dfrac{1}{(s+2)^2+3^2}$

$\frac{1}{2}\cdot 4$

$$= \frac{1}{3}\cdot\underbrace{\frac{3}{(s+2)^2+3^2}}_{\mathcal{L}[e^{-2t}\sin 3t]}$$

So $h(t) = \frac{1}{3} e^{-2t} \sin 3t$

Ans: $y(t) = \frac{1}{3} u_{\pi}(t)\, e^{-2(t-\pi)} \sin 3(t-\pi)$



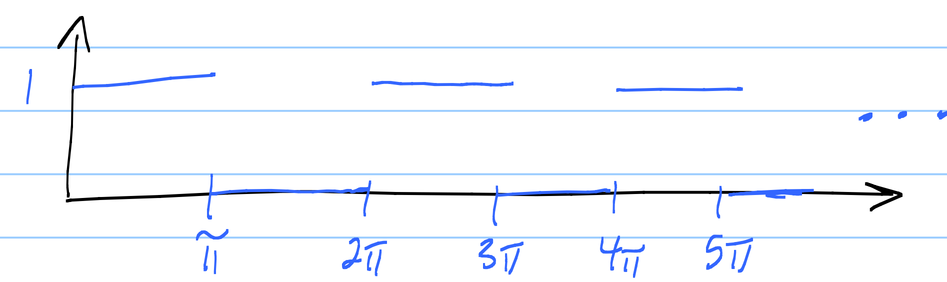
Answer in back of book: $e^{-2(t-\pi)} = e^{2\pi} e^{-2t}$

$\sin 3(t-\pi) = \sin(3t-3\pi)$

$= -\sin 3t$

EX: $y'' + y = g(t)$

$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$



$g(t) = [u_0(t) - u_{\pi}(t)] + [u_{2\pi}(t) - u_{3\pi}(t)] + [u_{4\pi}(t) - u_{5\pi}(t)] + \cdots$

$$G(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s} - \frac{e^{-3\pi s}}{s} + \frac{e^{-4\pi s}}{s} - \frac{e^{-5\pi s}}{s}$$

$$s^2 Y + Y = G(s)$$

$$Y(s) = \underbrace{\frac{1}{s(s^2+1)}}_{H(s) = \frac{1}{s} - \frac{s}{s^2+1}} \left[1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots \right]$$

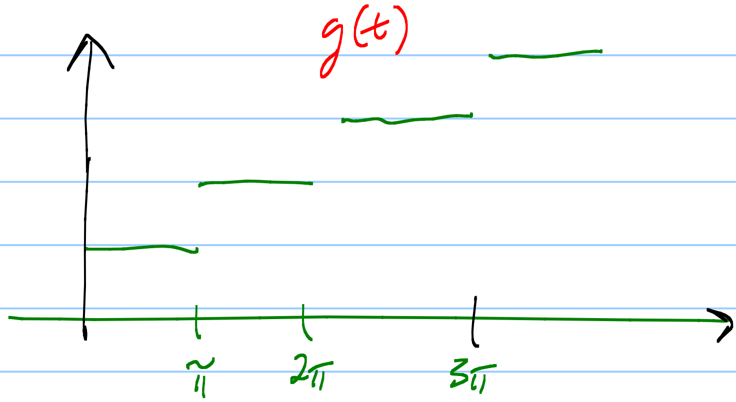
So $h(t) = 1 - \cos t$

$$y(t) = h(t) - u_\pi(t) h(t-\pi) + u_{2\pi}(t) h(t-2\pi) - \cdots$$

<u>Easy fact</u> $\cos(t - (\text{Even})\pi) = \cos t$

$\cos(t - (\text{ODD})\pi) = -\cos t$

$$y = \underbrace{\left(u_0 - u_\pi + u_{2\pi} - u_{3\pi} + \cdots \right)}_{g(t)} - \cos t \underbrace{\left(1 + u_\pi(t) + u_{2\pi}(t) + \cdots \right)}$$



Resonance!