Review lecture 1

Final Exam Thurs, Dec 16, 7:00–9:00pm in WTHR 200 Balcony ← door on third floor marked "Balcony 200"
(#2 pencils)

HW 36 due in MyLab tonight / HW33w, 34w, 35w, 36w due Thurs, 11:59 in GS

For Friday's review: Practice problems on home page

Linear first order ODEs

EX: $t^3 y' + t^2 y = t^{10}$

Step 1: Put in Standard Form:

$$y' + \underbrace{\frac{1}{t}}_{P(t)} y = t^7$$

Step 2: Get int. factor $\boxed{u = e^{\int P(t)\,dt}}$

$$= e^{\ln|t|} = |t| = \pm t$$

Take $u = t$.

Step 3: Use it:

$$u \cdot (\text{Standard Form eqn})$$

$$\underbrace{t\left(y' + \frac{1}{t} y\right)}_{\frac{d}{dt}[t y]} = t \cdot t^7 = t^8$$

So $\quad t y = \int t^8\,dt = \frac{1}{9} t^9 + C$

$$\boxed{y = \underbrace{\frac{1}{9}t^8}_{\text{Particular sol}^n} + \underbrace{C\cdot\frac{1}{t}}_{\text{Gen}^l\text{ Sol}^n\text{ to homog}}}$$

Separable equs: Ex:

$$\frac{dy}{dx} = y^3 + y^3e^{2x}$$

$$= y^3(1+e^{2x})$$

$$\int \frac{1}{y^3}\,dy = \int (1+e^{2x})\,dx$$

$$\frac{1}{-3+1}y^{-3+1} = x + \frac{1}{2}e^{2x} + C$$

Solve for y.

Homog equs: EX:

$$\frac{dy}{dx} = \frac{y^8 + x^3y^5}{xy^7}$$

$$= \left(\frac{y}{x}\right) + \left(\frac{x}{y}\right)^2$$

$$= \underbrace{\left(\frac{y}{x}\right) + \frac{1}{\left(\frac{y}{x}\right)^2}}_{v+\frac{1}{v^2}}$$

Let $\underbrace{v = \frac{y}{x}}$

Key: Solve for y:

$$y = xv$$

Take $\frac{d}{dx}$: $\frac{dy}{dx} = 1\cdot v + x\frac{dv}{dx} = x\frac{dv}{dx} + v$

Replaces $\frac{dy}{dx}$ on LHS.

New eqn: $$x\frac{dv}{dx} + v = v + \frac{1}{v^2}$$ ← Separable!

$$v^2\,dv = \frac{dx}{x}$$

$$\frac{1}{3}v^3 = \ln|x| + C$$

Solve for v: $$v = \sqrt[3]{3\ln|x| + 3C}$$

Finally, replace v by y/x and solve for y:

$$y = x\sqrt[3]{\ \ldots\ }$$

EXACT EQNS:

$$\frac{dy}{dx} = \frac{\text{eeeee}}{\text{mm}}$$

$$M + N\frac{dy}{dx} = 0$$

$$M\,dx + N\,dy = 0$$

Exactness test: $$\frac{\partial}{\partial y}\left[\text{Thing by } dx\right] = \frac{\partial}{\partial x}\left[\text{Thing by } dy\right]$$

$$\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$$

If yes, we can find a $\varphi(x,y)$ such that

$$\begin{cases} M = \dfrac{\partial \varphi}{\partial x} \\ N = \dfrac{\partial \varphi}{\partial y} \end{cases}$$

Remark: $M\,dx + N\,dy = \underbrace{\frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy}_{\text{exact differential}} = 0$

Sol^n to ODE: $\varphi(x,y) = C$

Method to solve if exact: $\begin{cases} \dfrac{\partial \varphi}{\partial x} = M & (A) \\[2ex] \dfrac{\partial \varphi}{\partial y} = N & (B) \end{cases}$

Step 1: Use (A):

$$\varphi = \underbrace{\int M(x,y)\,dx}_{m(x,y)} + \underbrace{C(y)}_{\substack{\text{Important!}\\ \text{Arb fcn of } y}}$$

Step 2: Use (B), but in a different way!

$$\frac{\partial}{\partial y}\left(\underbrace{m(x,y) + C(y)}_{\text{from (A) step}}\right) = N$$

$$\frac{\partial m}{\partial y}(x,y) + C'(y) = N(x,y)$$

Exactness guarantees that all the x's cancel out!

Get $C'(y) =$ fcn of y

So $C(y) = \int$ fcn of $y\ dy$

Step 3: Get $\varphi(x,y) = m(x,y) + C(y)$

Step 4: Set $\boxed{\varphi(x,y) = c}$ Gen' Sol^n

Step 5: Solve for y if you can

Initial value prob: $y(x_0) = y_0$

$$\varphi(x_0, y_0) \overset{\text{Def}^n}{=} c_0$$

Sol^n passing throug (x_0, y_0) is implicitly defined by

$$\varphi(x,y) = c_0.$$

Desperate changes of variables: EX: $\frac{dy}{dx} = (\underbrace{1+x+y}_{v})^{100}$

$$v = 1 + x + y$$

$$\frac{dv}{dx} = 0 + 1 + \frac{dy}{dx} \leftarrow \text{Aha! } \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = v^{100} \leftarrow \text{Separable!}$$

2nd Order ODEs:

Special non-linear ones: 1) $\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right) \leftarrow$ no x

Trick: Let $v = \frac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \underbrace{\frac{dy}{dx}}_{v}$$

$$\frac{d^2y}{dx^2} = v\frac{dv}{dy}$$

Alg: $\dfrac{d^2y}{dx^2} = f\left(y, \dfrac{dy}{dx}\right)$

$$v\frac{dv}{dy} = f(y, v) \leftarrow \text{1st order ODE}$$

Solve it. Get $v = (\text{fcn of } y)$

$\uparrow$

$\dfrac{dy}{dx} = (\text{fcn of } y) \leftarrow$ Separable!

2) $\dfrac{d^2y}{dx^2} = g\left(x, \dfrac{dy}{dx}\right) \leftarrow$ no y's $\quad v = \dfrac{dy}{dx}$

$$\frac{dv}{dx} = g(x, v) \leftarrow \text{first order ODE}$$

Solve it. Get $v = (\text{fcn of } x)$

$\uparrow$

$\dfrac{dy}{dx}$. Integrate to get y.

3) $\dfrac{d^2y}{dx^2} = h\left(x, y, \dfrac{dy}{dx}\right) \leftarrow$ ouch! (If not linear)

Linear 2nd order ODE with constant coeff

$$my'' + cy' + ky = F(t)$$

Homogeneous sol^n is Step 1.

Particular sol^n is Step 2.

Gen^l Sol^n: $\underbrace{(c_1 y_1 + c_2 y_2)}_{\text{homog sol}^{ns}} + y_p \leftarrow$ Part. sol^n.

Getting y_p : Use Undet Coeff if applicable.
(const coeff on left, special fcns on right)

If not, use Var of Par :

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Step 1: Solve homog eqn. Get $c_1 y_1 + c_2 y_2$

Step 2: Put in Standard Form :

$$y'' + \beta(x)y' + \gamma(x)y = Q(x)$$

Step 3: $y_p = u_1 y_1 + u_2 y_2$ where

$$\boxed{\begin{cases} u_1' = \dfrac{-y_2\, Q(x)}{W[y_1, y_2]} \\[2ex] u_2' = \dfrac{y_1\, Q(x)}{W[y_1, y_2]} \end{cases}}$$