More review

Office hours next week: M, T 2-3 pm MATH 750

Final exam: Wed, May 6, 8:00-10:00 am in Elliott Hall of Music,
20 question multiple choice exam covering whole course,
find Practice problems at www.math.purdue.edu/MA266

Prac prob 1-16 (Exm 1), 17-27 (Exm 2), 30-35 Laplace transf.
28-29 non-homog systems

Find answers on last page.

28. Find a particular solution of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{constant coeff}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\text{special thing}}.$$

A. $\mathbf{x}_p = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ B. $\mathbf{x}_p = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ C. $\mathbf{x}_p = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ D. $\mathbf{x}_p = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ✓ E. $\mathbf{x}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step 1: $\det \begin{bmatrix} 0-r & 1 \\ 1 & 0-r \end{bmatrix} = r^2 - 1 = (r-1)(r+1) = 0$

$r = \pm 1$

$\vec{x}_c = c_1 \vec{a}_1 e^{t} + c_2 \vec{a}_2 e^{-t}$

Safe to use undet. coeff: Try $\vec{x}_p = \vec{b}$

Plug in: $\vec{x}_p{}' = \vec{0} \overset{\text{want}}{=} A\vec{b} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{l} b_2 = 2 \\ b_1 = 3 \end{array}$$

$\vec{x}_p = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ D. ✓

29. Find the general solution of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 6e^{-t} \\ 1 \end{bmatrix}.$$

A. $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

B. $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

C. $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 6e^{-t} \\ 1 \end{bmatrix}$

D. $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

E. $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\det \begin{bmatrix} 2-r & 0 \\ 1 & 1-r \end{bmatrix} = (2-r)(1-r) = 0$

$r = \underline{1}, 2$

$\vec{F}_1(t) = \begin{pmatrix} -6 \\ 0 \end{pmatrix} e^{-t} \leftarrow$ safe!

$\vec{F}_2(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Split particular sol^n into two prob.

$\vec{x}_{p_1} = \vec{a}\, e^{-t}$ solves $\vec{x}_{p_1}' = \mathbb{A}\vec{x}_{p_1} + \vec{F}_1$

$\vec{x}_{p_2} = \vec{b}$ solves $\vec{x}_{p_2}' = \mathbb{A}\vec{b} + \vec{F}_2$ (like 28.)

$\vec{x}_p = \vec{x}_{p_1} + \vec{x}_{p_2}$

<u>Or</u> $\vec{x}_p = \vec{a}\, e^{-t} + \vec{b}$ plug in, collect terms, See 2 probs

Variation of parameters for systems

If rhs is "dangerous": pieces of non-homog term solve homog system.

$\vec{x}' = \mathbb{P}(t)\vec{x} + \vec{F}$.

<u>Step 1</u>. Get homog sol^n $\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2$.

$\mathbb{X} = [\vec{x}_1, \vec{x}_2]$

Part sol^n $\vec{x}_p = \mathbb{X}\,\vec{u}$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Condition: $\mathbb{X}\vec{u}' = \vec{F}$

$\vec{u}' = \mathbb{X}^{-1}\vec{F}$. Integrate to get u's.

30. $\mathcal{L}\{e^t(1+\cos 2t)\} = ?$ $\qquad$ $\mathcal{L}$ is linear!

A. $\frac{1}{s-1}+\frac{1}{(s-1)^2+4}$ $\quad$ B. $\frac{1}{s-1}\left(\frac{1}{s}+\frac{s-1}{(s-1)^2+4}\right)$ $\quad$ C. $\frac{1}{s-1}\frac{s-1}{s^2-2s+5}$ $\quad$ D. $\frac{1}{s}+\frac{s}{(s-1)^2+4}$

✓ E. $\frac{1}{s-1}+\frac{s-1}{s^2-2s+5}$

$$= \mathcal{L}[e^t] + \mathcal{L}[\underbrace{e^t \cos 2t}_{\text{s-shift rule}}]$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[\cos 2t] = \frac{s}{s^2+2^2}$$

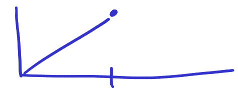
$$\mathcal{L}[e^t \cos 2t] = \frac{(s-1)}{(s-1)^2+2^2}$$

$$= \frac{1}{s-1} + \frac{(s-1)}{(s-1)^2+2^2} = \frac{1}{s-1} + \frac{s-1}{s^2-2s+5}$$

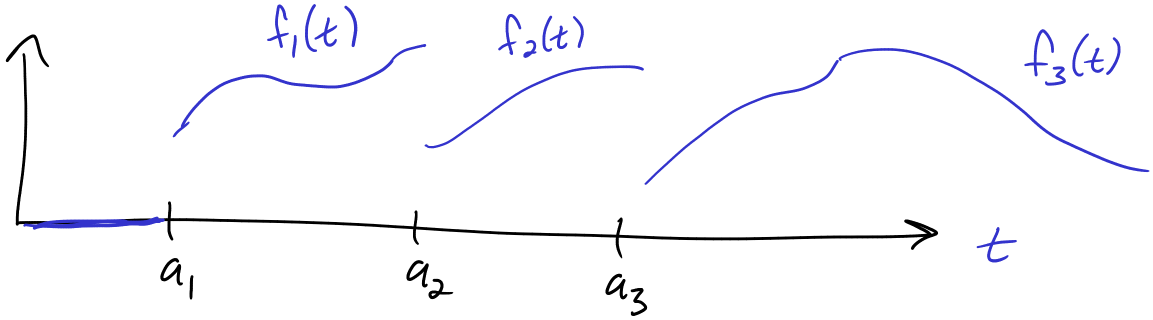
E. ✓

31. Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}.$$



A. $e^{-s}\left(\frac{1}{s} + \frac{1}{s-2}\right)$ B. $\frac{1}{s^2} - e^{-s}\frac{1}{s^2}$ C. $\frac{1}{s^2} - e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$ D. $\frac{1}{s^2} + 2e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$ E. $e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$



$$\left[u_{a_1}(t) - u_{a_2}(t)\right] \cdot f_1(t) + \left[u_{a_2}(t) - u_{a_3}(t)\right] \cdot f_2(t) + u_{a_3}(t) \cdot f_3(t)$$

on at a_1 off at a_2

$$f(t) = \left[u_0(t) - u_1(t)\right] \cdot t$$

$$\mathcal{L}\left[u_c(t) f(t-c)\right] = e^{-cs} F(s)$$

$$\underbrace{u_0(t)(t-0)}_{\text{same as fcn } t \text{ for } \mathcal{L}} - u_1(t)\, t$$

wish this were $(t-1)$

No problem! $u_1(t)\,t = u_1(t)\left[(t-1)+1\right]$

$$f(t) = t - u_1(t)\underbrace{(t-1)}_{\substack{g(t-1)=(t-1) \\ g(t)=t}} - u_1(t)$$

$$F(s) = \frac{1}{s^2} - e^{-s}\underbrace{\mathcal{L}(t)}_{\frac{1}{s^2}} - \frac{e^{-s}}{s} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

C. ✓

32. Solve

$$y'' + 3y' + 2y = 4u_1(t)$$
$$\underline{y(0) = 0}, \quad \underline{y'(0) = 1}.$$

~~A~~. $u_1(t)\left(2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right)$

B. $u_1(t)\left(2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right) + \underline{e^{-t} - e^{-2t}}$

~~C~~. $u_0(t)\left(2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right) + \underline{e^{-t} - e^{-2t}}$

~~D~~. $\left(2 - 4e^{-(t-1)} + 2e^{-2(t-1)}\right) + \underline{e^{-t} - e^{-2t}}$

~~E~~. $\underline{e^{-t} - e^{-2t}}$

careful!

$$\Big[s^2 Y(s) - s\cdot \underset{y(0)}{\underset{\uparrow}{0}} - \underset{y'(0)}{\underset{\uparrow}{1}}\Big] + 3\Big[sY(s) - \underset{y(0)}{\underset{\uparrow}{0}}\Big] + 2Y = 4\cdot\frac{e^{-s}}{s}$$

$$(s^2+3s+2)Y = 1 + 4\frac{e^{-s}}{s}$$

$$Y = \frac{1}{s^2+3s+2} + e^{-s}\underbrace{\frac{4}{s(s^2+3s+2)}}_{H(s)}$$

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} \quad \text{Get } A, B$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$$y_1 = Ae^{-2t} + Be^{-t} \sim \text{homog sol}^n$$

$$H(s) = \frac{1}{s(s+2)(s+1)} = \frac{a}{s} + \frac{b}{s+2} + \frac{c}{s+1}$$

Get a, b, c

$$h(t) = a + be^{-2t} + ce^{-t}$$

sol^n $\quad Ae^{-2t} + Be^{-t} + u_1(t)\,h(t-1)$

33. Find the solution of the initial value problem

$$y'' + y = \delta(t-\pi) \leftarrow \delta_{\pi}(t) \leftarrow \text{unit impulse, hits at } t=\pi$$
$$y(0) = 0, \quad y'(0) = 1.$$

$$\mathcal{L}[\delta_{\pi}(t)] = e^{-\pi s}$$

A. $y = \sin t + u_0(t)\sin(t-\pi)$ B. $y = \sin t + u_\pi(t)\sin(\pi t)$ C. $y = u_\pi(t)(\sin t + \sin(t-\pi))$ D. $y = u_\pi(t)\sin t$ E. $y = \sin t + u_\pi(t)\sin(t-\pi)$ ✓

$$(s^2 Y - s\cdot 0 - 1) + Y = e^{-\pi s} \qquad \mathcal{L}[u_c(t) f(t-c)] = e^{-cs}F(s)$$

$$Y = \frac{1}{s^2+1} + e^{-\pi s}\underbrace{\frac{1}{s^2+1}}_{F(s)}, \quad \text{So } f(t) = \sin t$$

$$\text{So} \quad y(t) = \sin t + u_{\pi}(t)\underbrace{\sin(t-\pi)}_{-\sin t}$$

$$\sin(t-2\pi) = \sin t$$

$$\cos(t-2\pi) = \cos t \qquad \cos(t-\pi) = -\cos t$$

34. The inverse Laplace transform of

$$F(s) = \frac{se^{-s}}{s^2 + 2s + 5}$$

is?

A. $u_1(t)\left(e^{t-1}\cos 2(t-1) - \frac{1}{2}e^{t-1}\sin 2(t-1)\right)$

B. $u_1(t)\left(e^{-t}\cos 2t\right) - \frac{1}{2}e^{-t}\sin 2t$

C. $u_1(t)\left(e^{-t+1}\cos 2(t-1) - \frac{1}{2}e^{-t+1}\sin 2(t-1)\right)$

D. $u_1(t)\left(e^{-t}\cos 2(t-1) - \frac{1}{2}e^{-t}\sin 2(t-1)\right)$

E. $e^{-t+1}\cos 2(t-1) - \frac{1}{2}e^{-t+1}\sin 2(t-1)$

$$\frac{as+b}{\text{Quadratic}} = \begin{cases} \frac{A}{s-r_1} + \frac{B}{s-r_2} & \text{if roots} \neq \text{, real.} \quad Ae^{r_1 t} + Be^{r_2 t} \\ \frac{A}{(s-r)^2} + \frac{B}{s-r} & \text{if real root repeated} \quad Ate^{rt} + Be^{rt} \\ \text{Complete square in denom} & \end{cases}$$

$$G(s) = \frac{s}{s^2+2s+5} = \frac{s}{(s+\frac{1}{2}\cdot 2)^2 + 4} = \frac{(s+1)-1}{(s+1)^2+2^2}$$

$\frac{1}{2}\cdot 2$

Table
$e^{-t}\sin 2t$
$e^{-t}\cos 2t$

$$= \frac{(s+1)}{(s+1)^2+2^2} - \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

$$= \mathcal{L}\Big[\underbrace{e^{-t}\cos 2t - \tfrac{1}{2}e^{-t}\sin 2t}_{g(t)}\Big]$$

$$F(s) = e^{-s}G(s) \qquad \mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s)$$

Ans: $f(t) = u_1(t)\,g(t-1)$

$$= u_1(t)\left[e^{-(t-1)}\cos 2(t-1) - \tfrac{1}{2}e^{-(t-1)}\sin 2(t-1)\right]$$

$$\overbrace{\quad}^{f(t-\tau)} \quad \overbrace{\quad}^{g(\tau)}$$

35. $\mathcal{L}\left\{\int_0^t \underbrace{\sin 2(t-\tau)}_{} \, \underbrace{\cos(3\tau)}_{} d\tau\right\} = ?$

$\mathcal{L}[f*g] = F(s)\,G(s)$

A. $\dfrac{1}{s^2+4}+\dfrac{s}{s^2+9}$ B. $\dfrac{2s}{(s^2+4)(s^2+9)}$ C. $\dfrac{2}{s^2+4}+\dfrac{s}{s^2+9}$ D. $\dfrac{2}{(s^2+4)(s^2+9)}$ E. $\dfrac{s}{(s^2+4)(s^2+9)}$

$$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

or

$f(t) = \sin 2t$ $F(s) = \dfrac{2}{s^2+4}$

$g(t) = \cos 3t$ $G(s) = \dfrac{s}{s^2+9}$

$F(s) \cdot G(s)$ ✓

18. In the phase portrait of the system

$$\det\begin{bmatrix} -3-r & -5 \\ 3 & 1-r \end{bmatrix}$$

$$\mathbf{x}' = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix}\mathbf{x},$$

the origin is a

A. asymptotically stable spiral point

B. saddle point

C. asymptotically stable node

D. asymptotically unstable node

E. asymptotically unstable spiral point

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ curve in plane

$r_1 < r_2 < 0$ $\quad e^{-t\text{'s}}$

A. Stable Improper Node

$0 < r_1 < r_2$ $\quad e^{+t\text{'s}}$

Unstable

$r = a \pm bi$

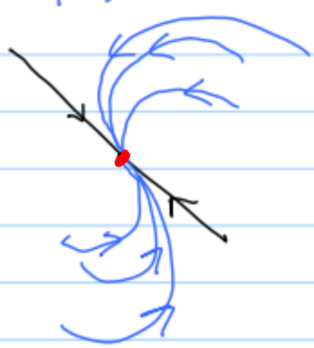
e^{at} (Cos and Sin bt)

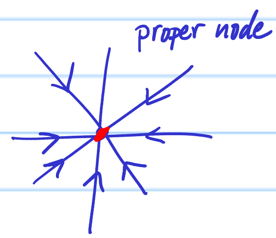
$a < 0$ Asym. Stable Spiral

$a > 0$ Unstable

Repeated roots:

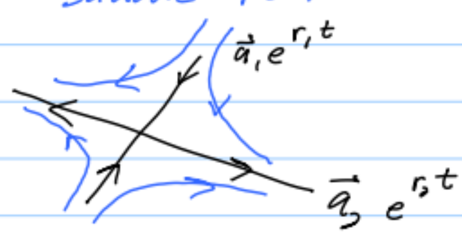
$r_1 = r_2 < 0$





$r_1 < 0 < r_2$

Unstable Saddle Point

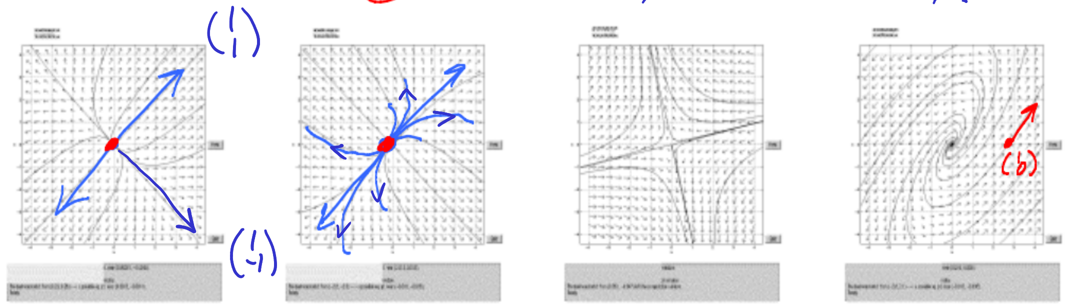


31. The phase portrait of the system $\vec{\mathbf{x}}'(t) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{\mathbf{x}}(t)$, whose general solution is

$\vec{\mathbf{x}}(t) = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, looks most like :

$r = 1, 3$

A. **B.** ~~**C.**~~ ~~**D.**~~



Node rule: Node. Sol'ns near origin hug the e.vect dir coresponding to the e.val closest to zero.

$r=1$

Reducing some second order ODE to first order

EX: $xy'' + 2y' = 6x$ ← y missing. Let $p = \frac{dy}{dx}$

Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

$x\frac{dp}{dx} + 2p = 6x$ ← separable

Solve for p. Then integrate: $y = \int p(x)\,dx$.

EX: $y\,y'' = (y')^2$ ← x missing. Let $p = \frac{dy}{dx}$

Important trick: Chain rule trick

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\underbrace{\frac{dy}{dx}}_{p} = p\frac{dp}{dy}$$

$$y\left(p\frac{dp}{dy}\right) = p^2$$ ← separable!

Get p as a fcn of y!

$$\underbrace{p}_{\frac{dy}{dx}} = G(y)$$ ← separable again!

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$$1\cdot y'' + P(x)y' + Q(x)y = 0$$

has two linearly independent sol^{n}s $y_1 = x$ and $y_2 = \frac{1}{x}$,

find a particular solution to

standard form ✓

$$1\cdot y'' + P(x)y' + Q(x)y = \frac{2}{x^2} \leftarrow F(x)$$

Know def^n of Wronskian $W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$

$$= \det\begin{bmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{bmatrix} = -\frac{2}{x}$$

Know $y_p = u_1 y_1 + u_2 y_2$ where

$$\begin{cases} u_1' = \dfrac{-y_2 F}{W} \\ u_2' = \dfrac{y_1 F}{W} \end{cases}$$

$\leftarrow F(x) = \frac{2}{x^2}$ from STANDARD FORM eqn

Prob: Can you figure out what $P(x)$ & $Q(x)$ are?

$$\begin{cases} y_1'' + Py_1' + Qy_1 = 0 \\ y_2'' + Py_2' + Qy_2 = 0 \end{cases}$$

Cramer's rule! Know W=det[ ] ≠ 0. Solve for P, Q. Yep! Bad at x=0.

$$\begin{bmatrix} y_1' & y_1 \\ y_2' & y_2 \end{bmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -y_1'' \\ -y_2'' \end{pmatrix} \quad !$$