Lecture 4 Separable equations (1.4)

HW 01,02,03 due in MyLab <u>tonight</u> 11:59 pm
Written HW 02,03,04 due in GS Friday 11:59 pm

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$\underbrace{g(y)\frac{dy}{dx}} = f(x)$$

$$\frac{d}{dx}\left[G(y)\right]$$

where $G(y)$ = anti der of $g(y)$

$$G(y) = \int g(y)\,dy$$

Chain rule

$$\frac{d}{dx}\left[(x^2+1)^5\right] = 5(x^2+1)^4(2x)$$

$$\frac{d}{dx}\left[y^5\right] = 5y^4\frac{dy}{dx}$$

$$\frac{d}{dx}G(y) = G'(y)\frac{dy}{dx}$$

Aha! If $G'(y) = g(y)$,
then RHS $= g(y)\frac{dy}{dx}$

So $G(y)$ = anti der of $f(x)$

$$G(y) = \underbrace{\int f(x)\,dx}_{F(x)} + C$$

$$\boxed{G(y) = F(x) + C}$$

Solve for y of x <u>if you can</u>.

Defines sol^n $y = y(x)$ <u>implicitly</u>.

How to remember method

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

<u>Step 1</u> Separate x's from y's.

$$g(y)\,dy = f(x)\,dx$$

Step 2 Integrate

$$\int g(y)\,dy = \int f(x)\,dx$$

$$\boxed{G(y) = F(x) + C}$$

only one $+C$. (Otherwise combine c_1, c_2 into one C.)

Step 3 Solve for y if you can.

Remark: IVP $y(x_0) = y_0$.

If you can't solve for y, get C from box: $G(y_0) = F(x_0) + C$.

EX $\dfrac{dy}{dx} = \dfrac{y}{x} = \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{y}\right)} = \dfrac{f(x)}{g(y)}$ ✓

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C$$

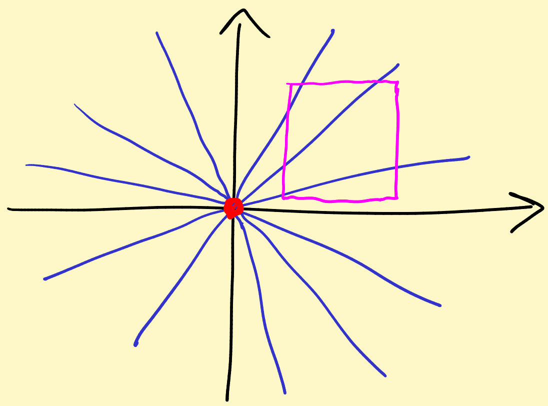
$$e^{\ln|y|} = e^{\ln|x| + C}$$

$$|y| = e^{\ln|x|}\, e^{C} = |x|\, e^{C}$$

$$\left|\frac{y}{x}\right| = e^{C}$$

$$\frac{y}{x} = \underbrace{\pm e^{C}}_{K}$$

$$\boxed{y = Kx}$$



$$\frac{dy}{dx} = \frac{y}{x} = f(x,y)$$

terrible when $x=0$

$y \equiv 0$ is a missing sol^{n}: $K = \lim_{C \to -\infty} (\pm e^{C}) = 0$

Note: Solutions don't cross (where and when $f(x,y)$ is well behaved: $f(x,y)$ cont. $\frac{\partial f}{\partial y}(x,y)$ cont.)

EX $\dfrac{dy}{dx} = e^{x+y}$ $= e^x e^y$ ✓

$$\boxed{\int e^{kx}dx = \frac{1}{k}e^{kx} + C}$$

$$\frac{1}{e^y}\,dy = e^x\,dx$$

$$\int e^{-y}\,dy = \int e^x\,dx$$

$$\frac{1}{(-1)}e^{-y} = e^x + C$$

$$e^{-y} = -e^x - C$$

Take Ln:

$$\ln e^{-y} = \ln\left(-e^x - C\right)$$

$$-y = \ln\Big[\underbrace{-C}_{K} - e^x\Big]$$

$C < 0$ for this to make sense.

$$\boxed{y = -\ln\left(K - e^x\right)}$$

Boom! when $x \to \ln K$

Feeling: First order ODEs give to rise to one constant C in solutions. (Because you do one integration to find solⁿ.)

Problem Given a "one parameter" family of curves, find an ODE that describes curves.

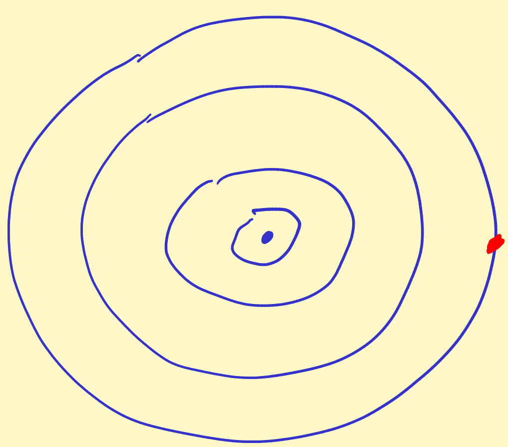
EX: $x^2 + y^2 = C$ ← circles, radius $\sqrt{C}$, $C \geq 0$,

y = unknown fcn of x

Implicit diff'tion:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$



← vertical tangent !

Hmmm. Solutions = functions of x.

Resolution: Think in terms of I.V.P.'s.

EX: $\frac{dy}{dx} = \frac{-x}{y}$ $\qquad y(1) = -2$

$$x^2 + y^2 = C$$

$$(1)^2 + (-2)^2 = C$$

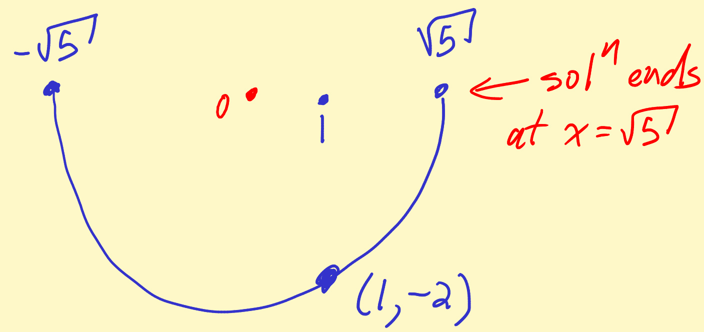
$$\boxed{C = 5}$$

$$x^2 + y^2 = 5$$

$$y = \pm\sqrt{5 - x^2}$$

$$y(1) = \pm\sqrt{5 - 1^2} = \pm 2$$ ← want -2 one

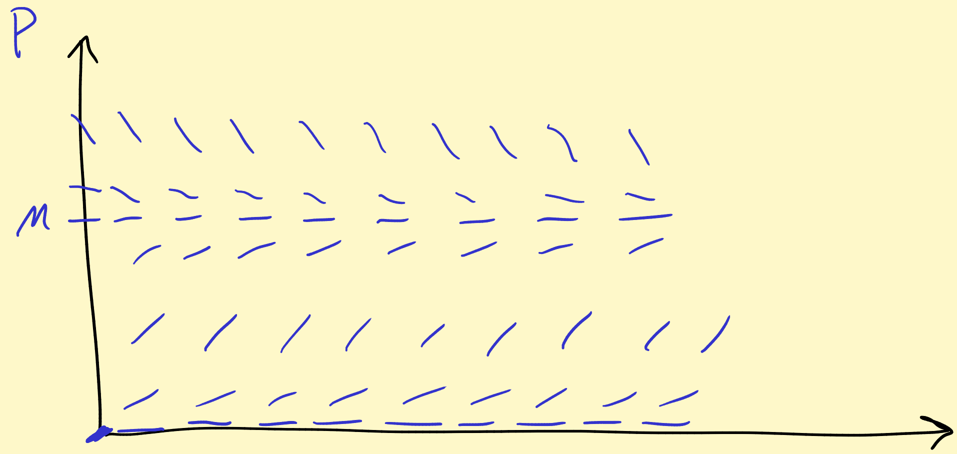
Ans $\quad y = -\sqrt{5 - x^2}$

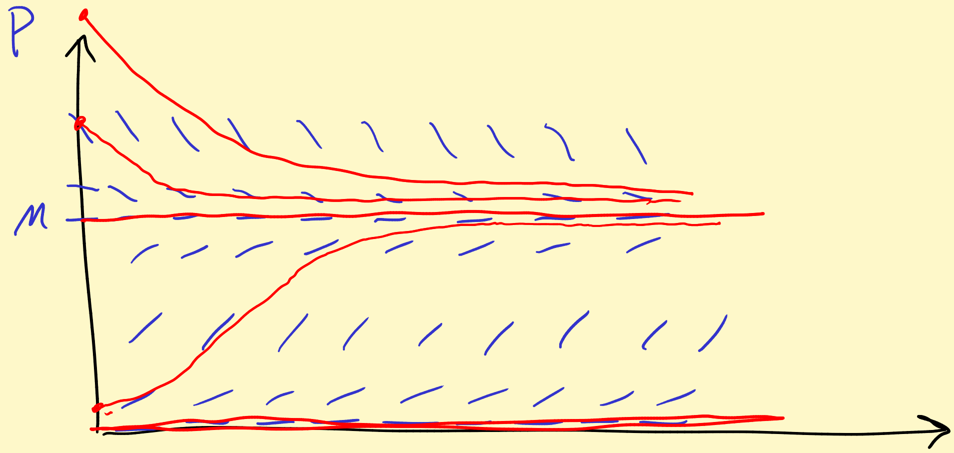


Logistic eqn

$$\frac{dP}{dt} = kP$$

$$\frac{dP}{dt} = \underbrace{\left[k(M-P)\right]}_{\text{improved "const"}} P$$





EX $\quad \dfrac{dy}{dx} = (2-y)\,y$

$$\int \frac{1}{(2-y)y}\,dy = \int 1\,dx = x + C$$

Partial fractions! $\quad \dfrac{1}{y(2-y)} = \dfrac{A}{y} + \dfrac{B}{2-y}$

Mult by denom $z(2-y)$: $\quad 1 = A(2-y) + By$

$$\underbrace{0}\cdot y + \underbrace{1} = \underbrace{(-A+B)}\,y + \underbrace{(2A)}$$

$=$ $\quad$ $=$

$$-A+B=0 \quad \longleftarrow \quad B=A=\tfrac{1}{2}$$

$$2A = 1 \quad \longleftarrow \quad A=\tfrac{1}{2}$$

$$\frac{1}{y(2-y)} = \frac{1/2}{y} + \frac{1/2}{2-y}$$

$$\int \frac{1}{y(2-y)}\,dy = \frac{1}{2}\int \frac{1}{y}\,dy + \frac{1}{2}\int \frac{1}{2-y}\,dy$$

$u = 2-y$

$du = -dy$

$$= \frac{1}{2}\ln|y| - \frac{1}{2}\ln|2-y|$$