Lecture 5 Linear equations (1.5)

{ ML HW04 due tonight
{ GS HW02,03,04 due tonight

$$A(x)\frac{dy}{dx} + B(x)\,y = D(x)$$

y terms are "linear"

$$\frac{dy}{dx} = \underbrace{\frac{-B(x)}{A(x)}\,y + \frac{D(x)}{A(x)}}_{f(x,y)}$$

See zeroes of $A(x)$ can be bad news.

EX $\quad x^2\frac{dy}{dx} + 2x\,y = x^3 \qquad A(x) = x^2 \leftarrow$ zero at $x=0$

Luck! LHS is $\frac{d}{dx}\left[x^2 y\right]$

So $x^2 y = \int x^3\,dx = \frac{1}{4}x^4 + C$

$$y = \frac{1}{4}x^2 + \frac{C}{x^2} \longleftarrow \text{nasty at } x=0 \checkmark$$

Method $\quad A(x)\frac{dy}{dx} + B(x)\,y = D(x)$

Step 1 Put in Standard form

Divide by $A(x)$: $\quad \boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$ *

where $P(x) = \frac{B(x)}{A(x)}$ and $Q(x) = \frac{D(x)}{A(x)}$

Step 2 Compute "integrating factor"

$$u(x) = e^{\int \underline{\underline{P(x)}}\,dx}$$

← no $+C$ here. Just want one u that works.

Step 3 Multiply the Standard form eqn by u:

$$\underbrace{u\frac{dy}{dx} + uPy} = uQ$$

← don't forget the u here

Step 4 LHS $= \frac{d}{dx}[uy]$

Step 5 Integrate: $uy = \int u(x)\,Q(x)\,dx + C$

yes to $+C$ here. Want all sol^{n}s.

Step 6 Get $y = \frac{1}{u(x)}\int u(x)Q(x)\,dx + \frac{C}{u(x)}$

Great to have a formula! See that where $P(x)$ and $Q(x)$ are cont., sol^n's exist and are unique.

Also great: Antiderivatives $\int P(x)\,dx = \int_{x_0}^{x} P(t)\,dt$

by Fund. Thm. Calc.

Get sol^{n}s via numerical integration when P, Q given by <u>data</u>.

<u>EX</u>: $$2\frac{dy}{dx} + y = e^{3x}$$

1) $$\frac{dy}{dx} + \underset{\uparrow P(x)}{\underline{\underline{\tfrac{1}{2}}}}\, y = \frac{1}{2}e^{3x}$$ Standard form

2) $$u = e^{\int \frac{1}{2}dx} = e^{x/2}$$ ← no $+C$

3) $$\underbrace{e^{x/2}\frac{dy}{dx} + \frac{1}{2}e^{x/2}y} = e^{x/2}\cdot\frac{1}{2}e^{3x} = \frac{1}{2}e^{7x/2}$$

4) $$\frac{d}{dx}\Big[\underset{\uparrow u}{e^{x/2}}\, y\Big]$$ ✓

5) $$e^{x/2}y = \int \frac{1}{2}e^{\frac{7}{2}x}\,dx + C$$

$$\left(\frac{e^{\frac{7}{2}x}}{e^{x/2}} = e^{\frac{6}{2}x} = e^{3x}\right) \qquad = \frac{1}{2}\cdot\frac{2}{7}e^{\frac{7}{2}x} + C$$

6) $$y = \frac{1}{7}e^{3x} + Ce^{-x/2} \leftarrow \text{Gen}^{l}\ \text{Sol}^{n}$$

[Better this way than trying to use formula.]

Find sol^n with $y(0) = 1$:

$$y(0) = \frac{1}{7}e^{3\cdot 0} + Ce^{0} \overset{\text{want}}{=} 1$$

$$\frac{1}{7} + C = 1 \qquad \boxed{C = 6/7}$$

Sol^n $$\boxed{y(x) = \frac{1}{7}e^{3x} + \frac{6}{7}e^{-x/2}}$$

<u>why choice of u</u> $$\frac{dy}{dx} + Py = Q$$

Mult by u: $$u\frac{dy}{dx} + \underline{uP}\,y = uQ$$

Wish: $$= \frac{d}{dx}\left[uy\right] = u\frac{dy}{dx} + \underline{u'}\,y$$

Aha! Need $u' = uP \leftarrow$ Separable eqn!

$$\frac{du}{dx} = uP(x)$$

$$\int \frac{1}{u}\,du = \int P(x)\,dx$$

$$\ln|u| = \int P(x)\,dx + C$$

Take e of:
$$e^{\ln|u|} = e^{\int P(x) + C}$$

$$|u| = e^{C}\, e^{\int P(x)\,dx}$$

$$u = \underbrace{(\pm e^{C})}_{K}\, e^{\int P(x)\,dx}$$

Just want one u that works. Take $K=1$. ✓

EX

$$\frac{dy}{dx} \underbrace{- (\operatorname{Tan} x)}_{P(x)}\, y = 8 \sin^3 x$$

Standard form ✓

$$u = e^{\int -\operatorname{Tan} x \, dx} = e^{-\ln|\operatorname{Sec} x|}$$

$$= e^{\ln|\operatorname{Sec} x|^{-1}}$$

$$= |\operatorname{Sec} x|^{-1}$$

$$= \left|\frac{1}{\operatorname{Cos} x}\right|^{-1}$$

$$= |\operatorname{Cos} x|$$

$$= \pm(\operatorname{Cos} x)$$

Take easy one $+\operatorname{Cos} x$

Get $\boxed{u = \cos x}$

$$\underbrace{\cos x \left(\frac{dy}{dx} - \tan x \; y \right)} = \left((\cos x) \, 8 \sin^3 x \right)$$

$$\frac{d}{dx}\left[((\cos x)) \; y \right]$$

u

$\checkmark$ $\tan x = \frac{\sin x}{\cos x}$

$\frac{d}{dx} \cos x = -\sin x$

$$(\cos x) y = \int \underbrace{8 \sin^3 x}_{8v^3} \; \underbrace{\cos x \; dx}_{dv}$$

$v = \sin x$

$dv = \cos x \, dx$

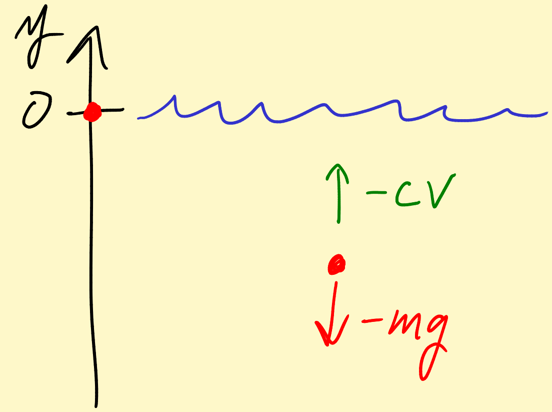
$$= \frac{8}{4} v^4 + C$$

$$= 2 (\sin x)^4 + C$$

$$y = 2 \frac{\sin^4 x}{\cos x} + \frac{C}{\cos x}$$

$$\boxed{y = 2 \tan x \sin^3 x + C \sec x}$$

Dropping rock in ocean



$ma = F$

$$\boxed{m\frac{dv}{dt} = -mg - cv}$$

Linear!

$$\frac{dv}{dt} + \underline{\underline{\frac{c}{m}}}\,v = -g$$

$\leftarrow$ Stand. form

$$u = e^{\int \frac{c}{m}\,dt} = e^{\frac{c}{m}t}$$

$$\underbrace{e^{\frac{c}{m}t}\left(\text{LHS}\right)} = -g e^{\frac{c}{m}t}$$

$$\frac{d}{dt}\left[e^{\frac{c}{m}t}\;v\right]$$

$$e^{\frac{c}{m}t}\,v = -\int g e^{\frac{c}{m}t}\,dt$$

$$= -\frac{mg}{c}e^{\frac{c}{m}t} + K$$

So $$v = -\frac{mg}{c} + K e^{-\frac{c}{m}t}$$

$\leftarrow$ $e^{-\frac{c}{m}t} \xrightarrow[t\to\infty]{} 0$

terminal velocity

Dropped from rest: $v(0) = 0$

want

$$v(0) = \frac{-mg}{c} + Ke^0 = 0$$

Get $\boxed{K = \frac{mg}{c}}$

Finally $\frac{dy}{dt} = v = -\frac{mg}{c} + \frac{mg}{c} e^{-\frac{c}{m}t}$

So $y = \int -\frac{mg}{c} + \frac{mg}{c} e^{-\frac{c}{m}t} \, dt$

$$= \frac{-mg}{c} t - \frac{m^2 g}{c^2} e^{-\frac{c}{m}t} + K_2$$

Need $y(0) = 0$. Get $K_2 = \frac{m^2 g}{c^2}$.

$$\boxed{y(t) = \frac{-mg}{c} t - \frac{m^2 g}{c^2} e^{-\frac{c}{m}t} + \frac{m^2 g}{c^2}}$$