Lecture 6 Applications

MyLab HW 05 due tonight

Newton's law of cooling

← hot rock in ocean

$T(t)$ = temp of rock at time t

A = temp of ocean

$$\frac{dT}{dt} \text{ prop to } (T(t) - A)$$

$$\boxed{\frac{dT}{dt} = -k\,(T - A)}$$

Linear! $$\frac{dT}{dt} + kT = kA$$

Good advice! If an eqn is linear, use linear method over others.

Standard form ✓ $$1 \cdot \frac{dT}{dt} + \underbrace{k}_{P(t)} T = kA$$

Int fact: $$u = e^{\int P(t)\,dt} = e^{\int k\,dt} = e^{kt}$$

e^{kt} [standard form eqn]:

$$\underbrace{e^{kt}\left[\frac{dT}{dt} + kT\right]}_{\frac{d}{dt}\left[e^{kt}\,T(t)\right]} = kA\,e^{kt}$$

!

$$e^{kt}\,T(t) = \int kAe^{kt}\,dt$$

$$= Ae^{kt} + C$$

$$T(t) = A + Ce^{-kt} \qquad \text{Gen}^{l}\ \text{Sol}^{n}$$

IVP: $T(0) = T_0$

$$\underbrace{T(0)}_{T_0} = A + C\underbrace{e^{0}}_{1}$$

So $\boxed{C = T_0 - A}$

Get $T(t) = A + (T_0 - A)e^{-kt}$

See $\lim\limits_{t\to\infty} T(t) = A$ no matter what T_0.

Prob 40° outside. Find stiff at 65° at time t_f. When was time of death?

$A = 40$. Let $t=0$ be at time of death.

$$T(t) = A + (T_0 - A)e^{-kt}$$

$$= 40 + (98.6^\circ - 40^\circ)e^{-kt}$$

Know $T(t_f) = 40 + 58.6 e^{-k t_F} = 65$

$$= e^{-k t_F} = \frac{25}{58.6}$$

Take Ln: $\quad \ln e^{-k t_F} = \ln 25/58.6$

$$-k t_F = \ln 25/58.6$$

Hmmm. Stuck. Only get $k t_F$. Want t_F.

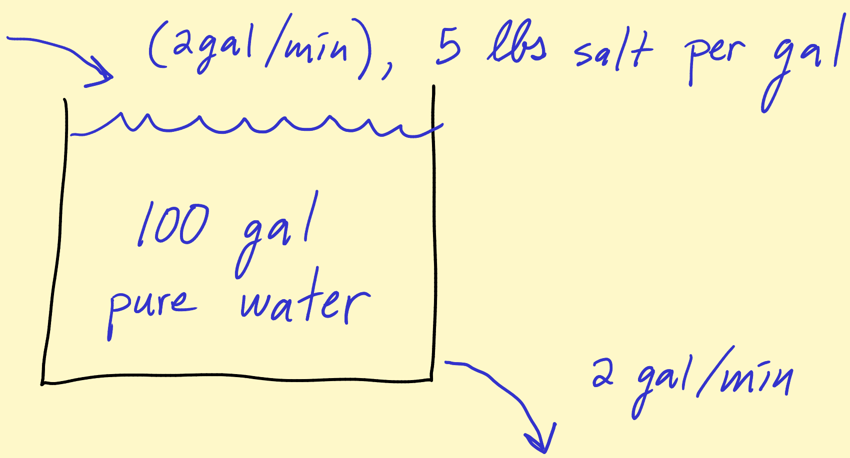
Aha! Wait an hour. Measure again. Get

$k(t_F + 1)$. $\quad \begin{cases} k t_F = \alpha \\ k(t_F+1) = \beta \end{cases}$ known $\quad$ 2 eqns, 2 unknowns

Solve for k and t_F.

Mixing problems

"well mixed"



How much salt in tank at time t (in minutes)?

Let $S(t)$ = lbs salt in tank at time t.

Hmmm. $\quad \frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$

$$= (2\ \text{gal}/\text{min})(5\ \text{lbs}/\text{gal}) - (2\ \text{gal}/\text{min})\left(\frac{S(t)}{100}\right)$$

Linear ODE!

$$\boxed{\frac{dS}{dt} = 10 - \frac{1}{50} S}$$

I.V.P. $S(0) = 0 \leftarrow$ pure at time 0

$$S' + \frac{1}{50} S = 10 \quad \text{Standard form} \checkmark$$

$$u = e^{\int \frac{1}{50} dt} = e^{t/50}$$

$$\left[e^{t/50} S\right]' = 10\, e^{t/50}$$

$$e^{t/50} S = 500\, e^{t/50} + C$$

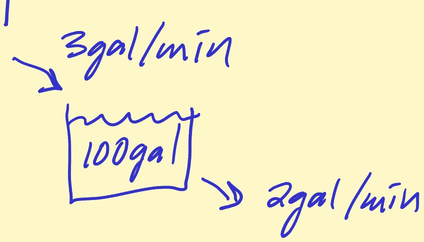
$$S = 500 + Ce^{-t/50}$$

Want $S(0) = 500 + Ce^{0} = 0$ ← want

Get $\boxed{C = -500}$

Ans: $\boxed{S(t) = 500 - 500e^{-t/50}}$

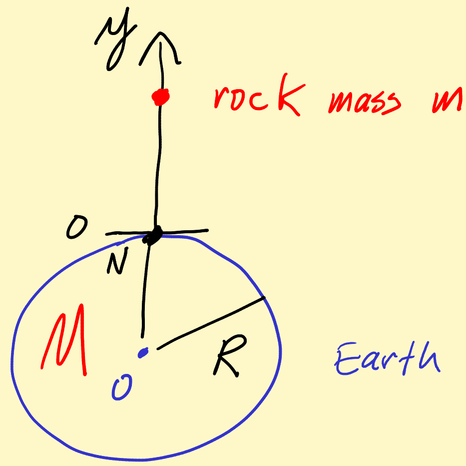
What if vol of tank is changing?



$$V(t) = 100 + (3-2)t = 100 + t$$

$$\text{Rate out} = \left(\frac{S(t)}{V(t)}\right)(2 \text{ gal/min})$$

Shoot a rock into space.



Newton's laws:

$$F = ma$$

$$\frac{-GMm}{(y+R)^2} = m\frac{dv}{dt}$$

Ouch! Too many variables: y, v, t

Classic trick: Chain rule: $\dfrac{dv}{dt} = \dfrac{dv}{dy}\underbrace{\dfrac{dy}{dt}}_{v}$

$$\boxed{\frac{-GMm}{(y+R)^2} = m\frac{dv}{dy} \cdot v}$$ ← separable!

$$\boxed{\frac{dv}{dt} = \frac{dv}{dy} \cdot v}$$

$$\int -\frac{GM}{(y+R)^2}\,dy = \int v\,dv$$

$$\frac{GM}{y+R} = \frac{1}{2}v^2 + C$$

I.V.P. Velocity at time 0 : $v(0) = v_0$

<u>And</u> : $y(0) = 0$

Plug in: $$\frac{GM}{0+R} = \frac{1}{2}v_0^2 + C$$

So $$\boxed{C = \frac{GM}{R} - \frac{1}{2}v_0^2}$$

$$\frac{GM}{y+R} = \frac{1}{2}v^2 + \left(\frac{GM}{R} - \frac{1}{2}v_0^2\right)$$

$$\boxed{y+R = \frac{GM}{\frac{1}{2}v^2 + \left(\frac{GM}{R} - \frac{1}{2}v_0^2\right)}}$$

Hmmm. Solve for v. Use $v = \frac{dy}{dt}$.

Get another (nasty) separable ODE!

How high does rock go? Aha! At max $y(t)$,

$v = \frac{dy}{dt} = 0$. Plug $v=0$ in.

Get $$y_{max} + R = \frac{GM}{\frac{1}{2}\cdot 0^2 + \left(\frac{GM}{R} - \frac{1}{2}v_0^2\right)}$$

Hmmm. Get y_{max} when $\frac{1}{2}v_0^2 < \frac{GM}{R}$.

Wow! as $v_0 \to \underbrace{\sqrt{\frac{2GM}{R}}}_{\text{escape velocity}}$, $y_{max} \to \infty$.

Fun to do. Go back, assume v_0 = escape velocity.

Nasty sep. eqn becomes an easy eqn. Get $y(t)$!

Fun things: $$\lim_{t\to\infty} y(t) = \infty$$

$$\lim_{t\to\infty} v(t) = 0$$

Slows down forever, looking like it is going to stop, but it goes forever and gets arbitrarily far away as $t \to \infty$!