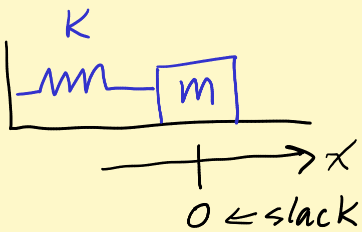
Lecture 7 Homogeneous eqns, Bernoulli eqns, Reduction of order (1.6)

Due tonight : MyLab HW 06
GS Written HW 05,06

Reduction of order

EX Work-Energy trick : $F = m\underbrace{a}_{\frac{d^2x}{dt^2}}$ ← 2nd order eqn



Hooke's $\underbrace{F}_{-kx} = m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\cdot\underbrace{\dfrac{dx}{dt}}_{v}$

$$\boxed{-kx = mv\frac{dv}{dx}}$$ ← separable! 1st order

No friction.

$$\int -kx\,dx = \int mv\,dv$$

$$-\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + C$$

Get work-energy eqn.

Left : potential energy
Right : kinetic energy

$$\Delta(\text{pot. eng.}) = \Delta(\text{kin. eng.})$$

Reduction of order $\dfrac{d^2y}{dx^2} = F\left(x, y, \dfrac{dy}{dx}\right)$

A) $$\underbrace{\frac{d^2y}{dx^2}}_{\frac{dp}{dx}} = F\Big(x, \underbrace{\frac{dy}{dx}}_{p}\Big)$$ ← y missing

$$\frac{dp}{dx} = F(x, p)$$ ← 1st order equ.

EX Drop rock in ocean $F = ma$

$$-mg - c\underbrace{\frac{dy}{dt}}_{v} = m\underbrace{\frac{d^2y}{dt^2}}_{\frac{dv}{dt}}$$

B) $$\underbrace{\frac{d^2y}{dx^2}}_{\frac{dp}{dx}} = F\Big(y, \underbrace{\frac{dy}{dx}}_{p}\Big)$$ ← x missing

Do work-energy trick: $\frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{p}$ Chain rule

$$\boxed{\frac{dp}{dx} = p\frac{dp}{dy}}$$

$$\frac{dp}{dx} = F(y, p)$$

$$p\frac{dp}{dy} = F(y, p)$$ ← 1st order ODE!

EX Shoot rock into space $F = ma$

$$-\frac{GMm}{(y+R)^2} = m\frac{d^2y}{dt^2} = m\frac{dv}{dt}$$

$$= m\,v\frac{dv}{dy}$$

Change of variables

EX $\dfrac{dy}{dx} = (\underbrace{x + y + 8}_{V})^2$

$V = x + y + 8$

y ← unknown fcn of x

So V = unknown fcn of x too

$\dfrac{dy}{dx} = V^2$ too many vars!

Key: Solve for y : $y = V - x - 8$

Take $\frac{dy}{dx}$ (use chain rule maybe)

$$\frac{dy}{dx} = \frac{dV}{dx} - 1 + 0$$

Plug into ODE

$$\left(\frac{dV}{dx} - 1\right) = V^2$$ ← separable!

$$\int \frac{dv}{1+v^2} = \int dx$$

$$\mathrm{Tan}^{-1} v = x + c$$

$$v = \mathrm{Tan}(x+c)$$

Last step. Get y. $\quad y = v - x - 8$

$$\boxed{y = \mathrm{Tan}(x+c) - x - 8}$$

Homogeneous eqns

$$\frac{dy}{dx} = F\Big(\underbrace{\frac{y}{x}}_{v}\Big)$$

$$v(x) = \frac{y(x)}{x} \leftarrow \text{unknown fcn of } x$$

$$\frac{dy}{dx} = F(v)$$

want this side in terms of $v, \frac{dv}{dx}, x$.

Key: Solve for y.

Take $\frac{dy}{dx}$.

$$v = \frac{y}{x}$$

$$\boxed{y = xv}$$

$$\frac{dy}{dx} = 1 \cdot v + x\frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

Plug into ode: $\quad \dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$

$$\boxed{v + x\frac{dv}{dx} = F(v)}$$

Separable!

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

the +C is inside h

Get $\underbrace{v}_{\frac{y}{x}} = $ fcn of x. $= h(x) \leftarrow$

$y = x \cdot h(x)$ done!

EX $\quad \dfrac{dy}{dx} = e^{y/x} + \dfrac{y}{x} \quad = F\left(\dfrac{y}{x}\right)$

$$v + x\frac{dv}{dx} = e^{v} + v$$

$$\int \frac{1}{e^{v}}\, dv = \int \frac{dx}{x}$$

$$\int e^{-v}\, dv = \ln|x| + C$$

$$\frac{1}{(-1)} e^{(-1)v} = \ln|x| + C$$

$$e^{-y/x} = -\ln|x| - C$$

Take Ln: $\underbrace{\ln e^{-y/x}}_{-\frac{y}{x}} = \ln\left[-\ln|x| - C\right]$

Solve for y: $y = -x \ln\left[-\ln|x| - C\right]$

EX: $\dfrac{x-y}{x+y} \cdot \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \dfrac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$ homog!

EX: $\ln y - \ln x = \ln \frac{y}{x}$ homog

Book $(x^2y^4 + y^6)\dfrac{dy}{dx} = (x^6 + x^2y^4 + x^3y^3)$

$$\frac{dy}{dx} = \frac{x^6 + x^2y^4 + x^3y^3}{x^2y^4 + y^6} \cdot \frac{\left(\frac{1}{x^6}\right)}{\left(\frac{1}{x^6}\right)}$$

See all total powers $= 6$

$$= \frac{1 + \left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^6} \quad \checkmark$$

Bernoulli eqn

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

Big idea: Change of var: $\boxed{v = y^{1-n}}$

Book: $y = v^{1/1-n}$

$$\frac{dy}{dx} = \frac{1}{1-n}\, v^{\frac{1}{1-n}-1} \cdot \frac{dv}{dx}$$

$$= \frac{1}{1-n}\, v^{\frac{n}{1-n}} \cdot \frac{dv}{dx}$$

Plug into ODE and get new eqn:

$$\boxed{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)}$$

Linear!

EX: Put in Standard form $\leftarrow$ Step 1

$$1 \cdot \frac{dy}{dx} - \frac{3}{2x}\,y = 2x\,y^{-1} \leftarrow y^n \qquad \boxed{n=-1}$$

$$v = y^{1-n} = y^{1-(-1)} = y^2$$

$P(x) = -\frac{3}{2x}$ $\qquad Q(x) = 2x$

New eqn:
$$\frac{dv}{dx} + (1 - \underbrace{(-1)}_{n})P(x)v = (1-(-1))Q(x)$$

$$\frac{dv}{dx} + 2\left(-\frac{3}{2x}\right)v = 2(2x)$$

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

Solve for v. $\quad v = y^2$, so $y = \pm\sqrt{v}$.