Lecture 8 Exact eqns (1.6)

Due tonight: MyLab HW 07

Bernoulli eqns: $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

Change of var: $v = y^{1-n} \leftarrow y = v^{1/(1-n)}$ [Bernoulli: Try $y = v^p$]

New linear eqn: $\boxed{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)}$

EX $\frac{dy}{dx} - \frac{3}{2x}y = 2x\,y^{-1} \leftarrow n=-1$

$v = y^{1-(-1)} = y^2$ $\boxed{v=y^2}$

New eqn: $\frac{dv}{dx} + (1-\underset{\substack{\uparrow \\ n=-1}}{(-1)})\underbrace{\left[-\frac{3}{2x}\right]}_{P(x)} v = (1-\underset{\substack{\uparrow \\ n}}{(-1)})\underbrace{\left[2x\right]}_{Q(x)}$

$$\frac{dv}{dx} - \frac{3}{x}v = 4x$$

$$u = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \frac{1}{x^3}$$

$$\underbrace{u\,[LHS]} = u \cdot 4x$$

$$\frac{d}{dx}\Big[\underset{\substack{\uparrow \\ u}}{\frac{1}{x^3}} \cdot v\Big] = \frac{1}{x^3}\cdot 4x = \frac{4}{x^2}$$

$$\frac{1}{x^3}v = \int \frac{4}{x^2}\,dx = -\frac{4}{x} + C$$

$$\underset{\substack{\parallel \\ y^2}}{V} = -4x^2 + Cx^3$$

$$y = \pm\sqrt{-4x^2 + Cx^3}$$

Choose C, $\pm$ via I.V.P.

<u>Exact equs</u> Solve ODE. Solve sol^n for C.

<u>EX</u> $x^2y + y^3 = C \leftarrow$ family of curves

Defines $y(x)$ <u>implicitly</u>.

Diff, use chain rule:

$$\frac{d}{dx}\left[x^2y\right] + \frac{d}{dx}y^3 = \frac{d}{dx}C$$

$$\left(2xy + x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\boxed{(2xy) + (x^2+3y^2)\frac{dy}{dx} = 0}$$

Want to go other direction!

<u>Chain rule</u> $\displaystyle \frac{d}{dx}\varphi(u,v) = \frac{\partial\varphi}{\partial u}\frac{du}{dx} + \frac{\partial\varphi}{\partial v}\frac{dv}{dx}$

Suppose $\varphi(x,y) = C$

$$\underset{u(x)=x}{\uparrow} \quad \underset{v(x)=y(x)}{\uparrow}$$

$$\frac{\partial \varphi}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

ODE : $\boxed{\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0}$

Hmmm. Given ODE : $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Can we cook up φ such that $M = \frac{\partial \varphi}{\partial x}$, $N = \frac{\partial \varphi}{\partial y}$.

If we can, then $\varphi(x,y) = C$ defines sol'ns to ODE!

Famous problem : Given a force field $\vec{F}$, when is $\vec{F}$ conservative, meaning work $\int_\gamma \vec{F}\cdot d\vec{s}$ is path independent?

MA 261 answer: On a rectangle, answer is :

1) When $\vec{F} = \nabla \varphi$ for some potential fcn φ, which is possible exactly when

2) Curl $\vec{F} = \vec{0}$ ⟸ 3D version

2D version in MA 261: $\vec{F} = M\hat{\imath} + N\hat{\jmath}$

$\exists\, \varphi$ with $\nabla\varphi = \vec{F}$ exactly when

$$\boxed{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0}$$

"2-D curl of $\vec{F}$"

$\vec{F} = M\hat{\imath} + N\hat{\jmath} + \underline{\underline{0}}\cdot\hat{k}$

Exactness criterion

$$\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

How to remember: $M + N\frac{dy}{dx} = 0$

Multiply by dx. $\boxed{\underbrace{M\,dx + N\,dy}_{\text{"differential 1-form"}} = 0}$

Advanced fact: $M\,dx + N\,dy$ is called <u>exact</u>

when it is equal to $d\varphi$ for some φ.

$$\underbrace{d\varphi = \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy}_{\text{total differential}}$$

Exactness criterion: $\frac{\partial}{\partial x}\underbrace{\left[\text{thing by } dy\right]}_{N} = \frac{\partial}{\partial y}\underbrace{\left[\text{thing by } dx\right]}_{M}$

Easy direction: If $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$, then

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial y}\right] \overset{?}{=} \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial \varphi}{\partial x}\right]$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x} \leftarrow \text{mixed partials equal} \checkmark$$

Other direction: MA 261 proof. [Book does it.]

<u>How to find φ</u> <u>Ex</u>

$$\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}\frac{dy}{dx} = 0$$

Exact?

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2xy + \sin y\right) = 2y + 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^2 + \cos x\right) = 2y + 0$$

$= \checkmark$ yes!

Can cook up φ with

$$\frac{\partial \varphi}{\partial x} = M = y^2 + \cos x \quad (A)$$

$$\frac{\partial \varphi}{\partial y} = N = 2xy + \sin y \quad (B)$$

<u>Step 1</u>: Use (A):

$$\varphi = \int y^2 + \cos x \, dx$$

"partial integral"
Treat y like a constant.

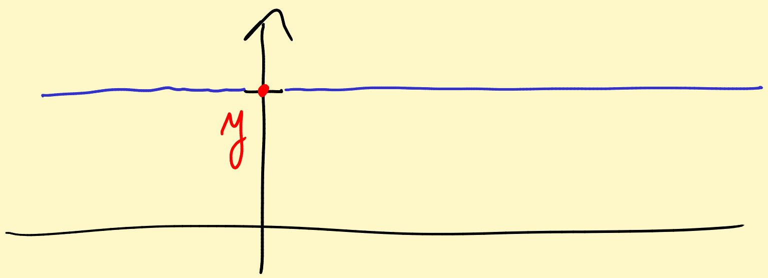
$$= xy^2 + \sin x + \underbrace{C(y)}$$

unknown fcn of y in place of + constant.

Why $C(y)$ instead of just C:

If φ_1 and φ_2 both satisfied $\frac{\partial \varphi_i}{\partial x} = M$, then

$$\frac{\partial}{\partial x}\left(\varphi_1 - \varphi_2\right) \equiv 0$$



$\varphi_1 - \varphi_2$ is constant on horizontal lines.

Could be different on different lines.

Also check $\frac{\partial}{\partial x}\left[\varphi + C(y)\right] = \frac{\partial \varphi}{\partial x}$ ✓

(A) is used up. Got

$$\boxed{\varphi = xy^2 + \sin x + C(y)}$$

Step 2 Use (B) (in a different way)

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\Big[\, xy^2 + \sin x \;\; + \underbrace{C(y)}_{\text{from (A)}} \Big]$$

$$\overset{\text{want}}{=} \underbrace{2xy + \sin y}_{N}$$

$$2xy + 0 + C'(y) \overset{\text{want}}{=} 2xy + \sin y$$

Step 3 Confirm miracle: All x's cancel out!

$$\boxed{C'(y) = \sin y}$$

Step 4 Get $C(y) = \int \sin y \, dy = -\cos y + K$

don't need $+K$ here

Step 5 Write $\varphi(x,y)$

$$xy^2 + \sin x \;\; + \big(-\cos y + k\big)$$

Step 6 Answer: $xy^2 + \sin x - \cos y + K = C$

Defines sol^{n}s $y(x)$ implicitly.

(See can absorb $+K$ into C)

<u>Step 7</u> Solve for y if you can.

Note: Can always get C from I.V.P $y(x_0) = y_0$. Plug in (x_0, y_0). Spit out C.