Lecture 9 Population dynamics (2.1)

MyLab HW 08 due tonight

Exact equns: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Write as $M\,dx + N\,dy = 0$

Check $\frac{\partial}{\partial x}\underbrace{\left[\begin{matrix}\text{Thing by}\\ dy\end{matrix}\right]}_{N} \overset{?}{=} \frac{\partial}{\partial y}\underbrace{\left[\begin{matrix}\text{Thing by}\\ dx\end{matrix}\right]}_{M}$

Yes, then exact. Can find φ with $\nabla\varphi = M\hat{i} + N\hat{j}$,

i.e., $\frac{\partial\varphi}{\partial x} = M$ and $\frac{\partial\varphi}{\partial y} = N$.

Gen^l sol^n is $\boxed{\varphi(x,y) = C}$ ← defines y implicitly

EX $\underbrace{(ye^{xy} + 2x)}_{M} + \underbrace{(xe^{xy} - y)}_{N}\frac{dy}{dx} = 0$

Test: $\frac{\partial}{\partial x}\Big[\underbrace{xe^{xy} - y}_{N}\Big] \overset{?}{=} \frac{\partial}{\partial y}\Big[\underbrace{ye^{xy} + 2x}_{M}\Big]$

$$1\cdot e^{xy} + x(ye^{xy}) - 0 \overset{?}{=} 1\cdot e^{xy} + y(xe^{xy}) + 0$$

✓ exact!

Cook up φ with $\begin{cases} \frac{\partial\varphi}{\partial x} = M = ye^{xy} + 2x & \text{(A)} \\ \frac{\partial\varphi}{\partial y} = N = xe^{xy} - y & \text{(B)} \end{cases}$

First use (A): $\varphi = \int ye^{xy} + 2x \, dx$ ← treat y like a constant

$$= y\left[\frac{1}{y}e^{xy}\right] + x^2 + \underbrace{C(y)}_{\text{arb fcn of } y}$$

(A) is used up.

$$= \underbrace{e^{xy} + x^2 + C(y)}_{\varphi(x,y)}$$

Next use (B) (in a different way!)

$$(B): \quad \frac{\partial \varphi}{\partial y} = N = xe^{xy} - y$$

$$\frac{\partial}{\partial y}\left[\underbrace{e^{xy} + x^2 + C(y)}_{\text{from (A)}}\right] \overset{\text{want}}{=} xe^{xy} - y$$

$$xe^{xy} + 0 + C'(y) \overset{\text{want}}{=} xe^{xy} - y$$

Miracle! x's cancel: $\boxed{C'(y) = -y}$

$$C(y) = \int -y\, dy = -\frac{1}{2}y^2 \qquad \text{(skip } + k\text{)}$$

Get $\varphi = e^{xy} + x^2 + \underbrace{\left(-\frac{1}{2}y^2\right)}_{C(y)} = C$

Populations dynamics: $\frac{dp}{dt} = kp$ Malthusian

Logistic eqn: $$\frac{dp}{dt} = \left[k(M-p)\right] p$$

Separable eqn: $$\underbrace{\int \frac{dp}{p(M-p)}}_{\text{partial fractions}} = \int k\,dt = kt + C$$

$$\frac{1}{p(M-p)} = \frac{A}{p} + \frac{B}{M-p}$$

Mult by denom $p(M-p)$: $$1 = A(M-p) + Bp$$

$$0 \cdot p + 1 = \underbrace{(B-A)}p + \underbrace{(AM)}$$

must =

=

$$\begin{cases} AM = 1 \\ B - A = 0 \end{cases} \qquad \begin{aligned} A &= 1/M \\ B = A &= 1/M \end{aligned} \qquad \boxed{\begin{aligned} A &= 1/M \\ B &= 1/M \end{aligned}}$$

$$\int \frac{1}{P(M-P)}\,dP = \int \frac{1/M}{P} + \frac{1/M}{\underbrace{M-P}_{u}}\,dP \qquad du = -dP$$

$$= \frac{1}{M} \ln P - \frac{1}{M} \ln (M-P) = kt + C$$

$$\ln P - \ln(M-P) = kMt + CM$$

$$\ln \frac{P}{M-P} = kMt + CM$$

$$\frac{P}{M-P} = e^{kMt} \underbrace{e^{CM}}_{c} = c e^{kMt}$$

$$P = Mce^{kMt} - ce^{kMt} P$$

$$P \left[1 + ce^{kMt} \right] = Mce^{kMt}$$

$$P = \frac{Mce^{kMt}}{1 + ce^{kMt}} \cdot \frac{e^{-kMt}}{e^{-kMt}}$$

$$P = \frac{Mc}{c + e^{-kMt}}$$

See $\lim_{t \to \infty} P(t) = \frac{Mc}{c} = M$

Remark Can also solve as a Bernoulli eqn!

$$\frac{dP}{dt} = k(M-P)P = kMP - kP^2$$

$$\frac{dP}{dt} - kM\,P = -k\,P^{2} \leftarrow n=2$$

Bernoulli: $V = P^{1-n} = P^{1-2} = \dfrac{1}{P}$.

Do it.

IVP: $P(0) = P_0$

$$P(t) = \frac{Mc}{c + e^{-kMt}}$$

want

$$P(0) = \frac{Mc}{c + e^{kM\cdot 0}} = \frac{Mc}{c+1} = P_0$$

Solve for c: $c = \dfrac{P_0}{M-P_0}$

Get $\boxed{P(t) = \dfrac{P_0 M}{P_0 + (M-P_0)e^{-kMt}}}$

Solution to Logistics equ with $P(0) = P_0$

Autonomous first order ODE's :

$$\frac{dP}{dt} = f(P) \leftarrow \text{no time on RHS}$$

Direction fields are easy!

$$\frac{dP}{dt} = K(M-P)P \leftarrow \quad = f(P)$$

KP: $- - - - -$ $\underset{0}{\overset{0}{|}}$ $+ + + + + + + + + + + + + + + +$

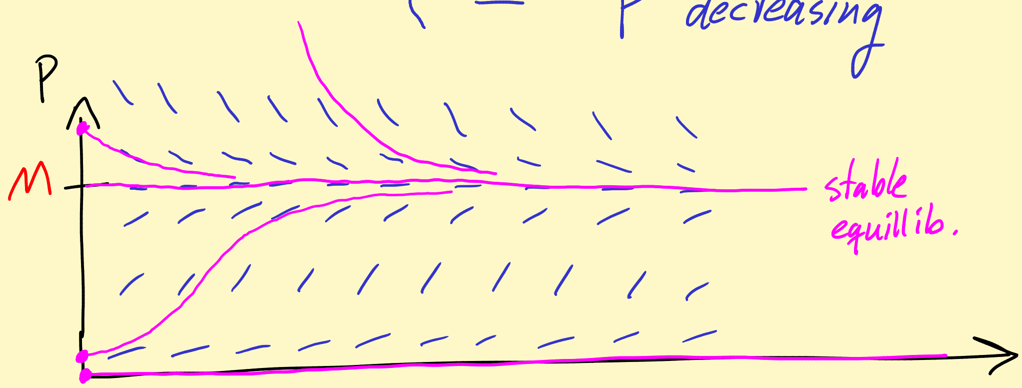
$M-P$: $+ + + + + + + + + + + + + + + +$ $\underset{M}{\overset{0}{|}}$ $- - - - - -$

Sign of $f(P)$: $K(M-P)P$: $- - - - - - -$ $\overset{0}{|}$ $+ + + + + + + + + + + +$ $\overset{0}{|}$ $- - - - - - -$

$\leftarrow 0 \rightarrow \qquad \rightarrow M \leftarrow$

$$\frac{dP}{dt} = f(P) = \begin{cases} + & P \text{ increasing} \\ 0 & P \text{ flat} \leftarrow \text{constant sol}^n\text{s where } f(P)=0 \\ - & P \text{ decreasing} \end{cases}$$

equillibrium sol^{n}s



Autonomous: slope field only depends on P: same slope along horiz lines

Fun thing: Where are inflection pts in $0 < P < M$?

Pts where $\frac{d^2P}{dt^2}$ changes sign

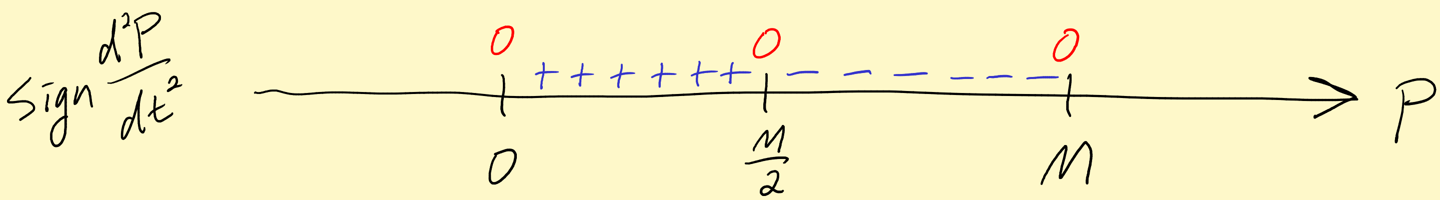
$$\frac{dP}{dt} = k(M-P)P = kMP - kP^2$$

$$\frac{d^2P}{dt^2} = kM\frac{dP}{dt} - k\,2P\frac{dP}{dt}$$

$\frac{dP}{dt} = k(M-P)P$

$$= (kM - k2P)\frac{dP}{dt}$$

$$= k(M-2P)\,k(M-P)P$$



Inflection when sol^{n}s pass through $P = \frac{M}{2}$