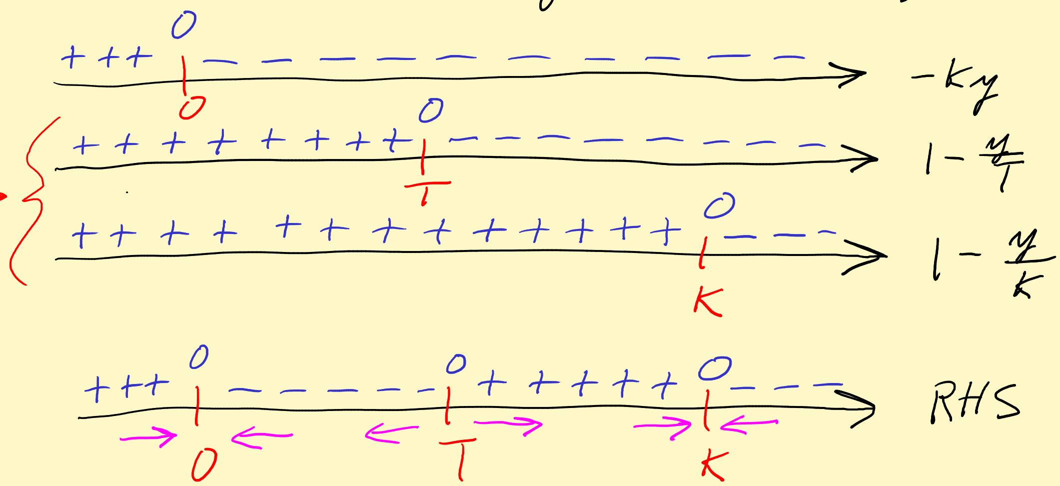
Lecture 11 Mechanics (2.3) MyLab HW 10

Last time: Carrier pigeons $\frac{dy}{dt} = -ky\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)$

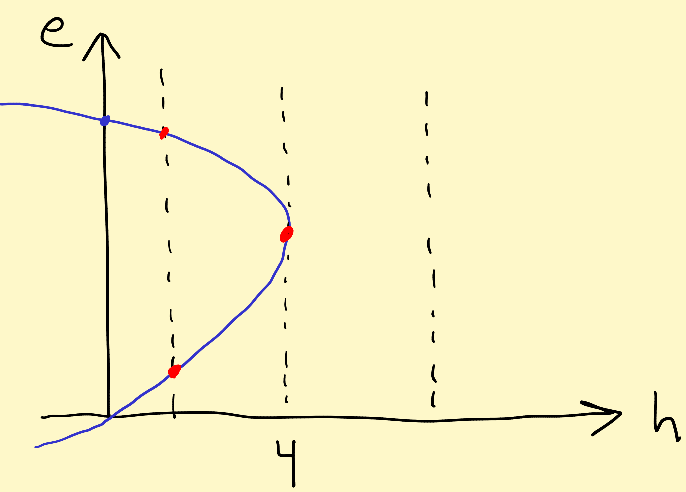
Correct $\pm$'s (Fixed in Lecture 10 notes)



Bifurcation diagram

EX $\frac{dy}{dt} = \underbrace{y(4-y)-h}$

equillibrium sol^{n}s
$y \equiv e$, e root of RHS



quadratic in e $\rightarrow$ $e(4-e)-h = 0$

Get $e = 2 \pm \sqrt{4-h}$

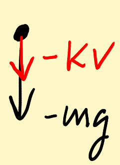
↑ bifurcation pt = 4

Book: Uses c in place of e. "critical values" are the same as the e's in equil. sol^{n}s.

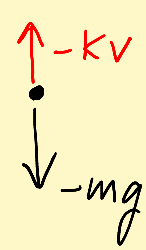
EX Stokes' law: Friction force $= -kv$

Acurate for "laminar flow" ← slow and viscous fluid

Up: $v \uparrow$



Down: $v \downarrow$



$$F = ma$$

$$-mg - kv = m\frac{dv}{dt}$$ ← first order linear ODE with const coeff!

Got: Terminal velocity: Happens when: Going down

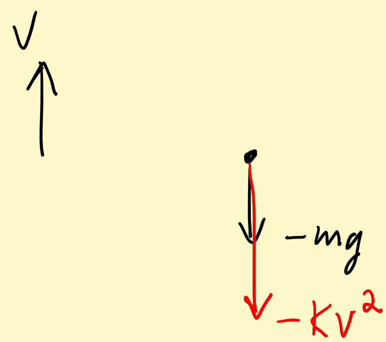
friction force = weight

$$|-kv| = |-mg|$$

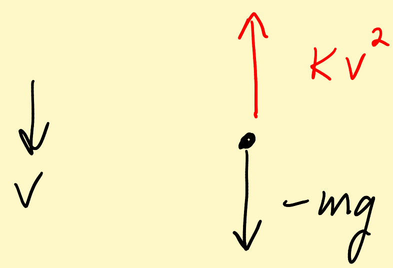
going down $v < 0$ so $v = -mg/k$ → $|v| = \frac{mg}{k}$

Higher velocity motion with turbulence: Friction $\propto v^2$

Up:



Down:



Two problems!

Up: $F = ma$

$$-mg - kv^2 = m\frac{dv}{dt}$$ ← separable

$$\int \frac{m\,dv}{mg + kv^2} = \int -dt$$

Table $\int \frac{1}{1+u^2}\,du = \operatorname{Tan}^{-1} u + C$

$$\frac{1}{mg}\int \frac{m\,dv}{1+\frac{k}{mg}v^2} = -t + C$$

Aha! $u = \sqrt{\frac{k}{mg}}\cdot v \qquad du = \sqrt{\frac{k}{mg}}\,dv$

$$dv = \sqrt{\frac{mg}{k}}\,du$$

$$\frac{1}{g}\int \frac{\overbrace{\sqrt{\frac{mg}{k}}\,du}^{dv}}{1+u^2} = -t + C$$

$$\sqrt{\frac{m}{gk}}\ \mathrm{Tan}^{-1}\Big(\underbrace{\sqrt{\frac{k}{mg}}\,v}_{u}\Big) = -t + C$$

Use initial cond $v = v_0$ at time $t = 0$ to get C here.

Continue. Want $v =$ fcn of t.

$$\mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = -\sqrt{\frac{gk}{m}}\,t + \sqrt{\frac{gk}{m}}\cdot C$$

$$v = \sqrt{\frac{mg}{k}}\ \mathrm{Tan}\left(\sqrt{\frac{gk}{m}}\cdot C - \sqrt{\frac{gk}{m}}\,t\right)$$

Fun: $v = 0$ when bullet peaks. Get t_{max}.

Continue: $\quad v = \frac{dy}{dt}$

Ouch! $\quad y = \int (\quad)\,dt$

Use $$\int \tan u \, du = \ln|\sec u| + C$$

<u>Down</u> : $F = ma$

$$-mg + kv^2 = m\frac{dv}{dt}$$

$$\int \frac{dv}{g - \frac{k}{m}v^2} = \int -dt = -t + C$$

Use $$\int \frac{du}{1-u^2} = \tanh^{-1} u + C$$

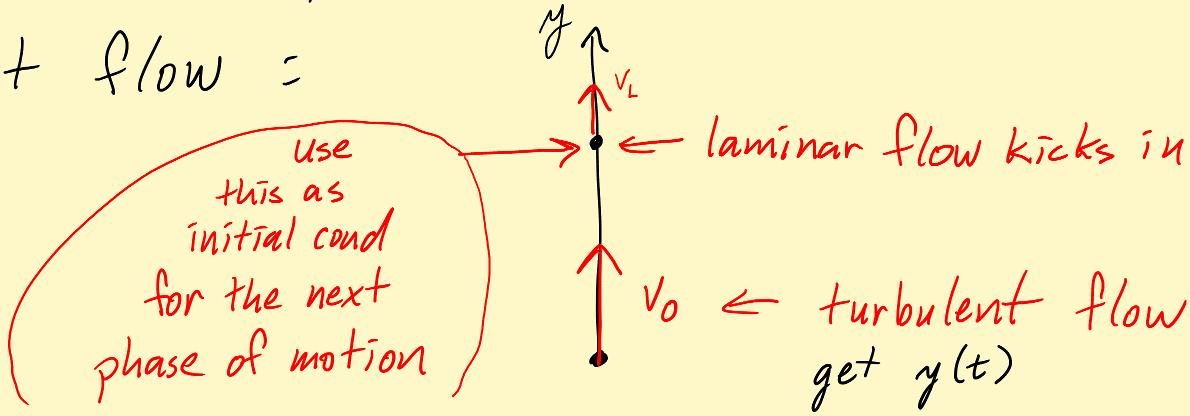
and $$\int \tanh^{-1} u \, du = \ln(\cosh u) + C$$

<u>Cheat</u>: What is terminal velocity? $mg = kv^2$

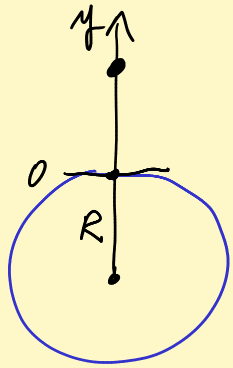
$$|v| = \sqrt{\frac{mg}{k}}$$

$$v = -\sqrt{\frac{mg}{k}}$$

<u>Real life</u> : Boundary between laminar flow and turbulent flow :



Shoot rock into space



$$F = ma$$

$$\frac{-GMm}{(y+R)^2} = m\frac{dv}{dt}$$

Trick :
$$\frac{dv}{dt} = \frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v} = v\frac{dv}{dy}$$

$$\frac{-GMm}{(y+R)^2} = mv\frac{dv}{dy}$$

$$\int -\frac{GMm}{(y+R)^2}\,dy = \int mv\,dv$$

Physics

$$\underbrace{\int_0^y -\frac{GMm}{(y+R)^2}\,dy}_{\Delta(\text{Potential energy})} = \int_{v_0}^{v} mv\,dv$$

$$\Delta(\text{Potential energy}) = \Delta\left(\tfrac{1}{2}mv^2\right)$$

Kinetic energy

Math

$$\int (mm)\,dy = \frac{GMm}{(y+R)} = \frac{1}{2}mv^2 + C$$

Did this in Lecture 6.

$$V^2 = \frac{2GM}{(y+R)} - \frac{2GM}{R} + V_0^2$$

$$\underset{\displaystyle \frac{dy}{dt}}{\underset{\uparrow}{V}} = \sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + V_0^2}$$

Get
$$\int \underbrace{\frac{dy}{\sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + V_0^2}}} = \int dt$$

$$= t + C$$

Big hairy fcn of y!

Must solve for $y =$ fcn of t numerically.

Nice case: $V_0 =$ escape velocity $\sqrt{\frac{2GM}{R}}$

Get
$$\int \frac{dy}{\sqrt{\frac{2GM}{(y+R)}}} = t + C$$

$$\frac{1}{\sqrt{2GM}} \int \underbrace{(y+R)^{1/2}}_{\frac{1}{\frac{1}{2}+1}(y+R)^{\frac{1}{2}+1}} \, dy = t + C$$

Get $$\frac{1}{\sqrt{2GM}} \cdot \frac{2}{3} (y+R)^{3/2} = t + C$$

$y = 0$ when $t = 0$

$$C = \frac{2R^{3/2}}{3\sqrt{2GM}}$$

$$\boxed{y = \left(\frac{3}{2} \sqrt{2GM} \cdot (t + C) \right)^{2/3}}$$

$\to \infty$ as $t \to \infty$

$$\frac{dy}{dt} = \sqrt{2GM} \, (t + c)^{-1/3}$$

$\to 0$ as $t \to \infty$