Lecture 12 Numerical methods (2.4)

MyLab HW 11

Last time: Shoot rock into space.

Got $V = \sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + V_0^2}$

$v = dy/dt$

$$\int \frac{dy}{\sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + V_0^2}} = \int dt = t + C$$

Case $V_0 = \sqrt{\frac{2GM}{R}}$ escape velocity.

Get
$$\int \frac{dy}{\sqrt{\frac{2GM}{(y+R)}}} = t + C$$

$$\frac{1}{\sqrt{2GM}} \underbrace{\int (y+R)^{1/2}\, dy}_{\frac{1}{\frac{1}{2}+1}(y+R)^{\frac{1}{2}+1}} = t + C$$

Get
$$\frac{1}{\sqrt{2GM}} \cdot \frac{2}{3} (y+R)^{3/2} = t + C$$

$y = 0$ when $t = 0$

$C = \frac{2R^{3/2}}{3\sqrt{2GM}}$

$$\boxed{y = \left(\frac{3}{2}\sqrt{2GM}\cdot t + R^{3/2}\right)^{2/3}}$$

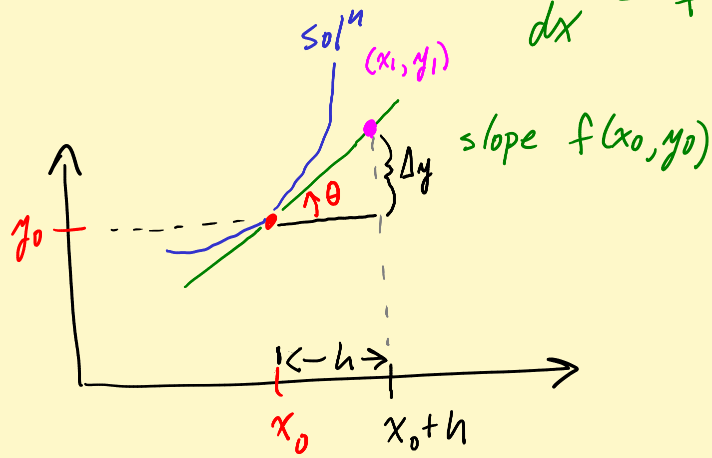
$\to \infty$ as $t \to \infty$

$$\frac{dy}{dt} = \sqrt{2GM}\,(t + c)^{-1/3}$$

$\to 0$ as $t \to \infty$

Euler's method

$$\frac{dy}{dx} = f(x,y) \quad , \quad f(x_0) = y_0$$



$$\begin{cases} x_1 = x_0 + h \\ y_1 = y_0 + h f(x_0, y_0) \end{cases}$$

$$\text{slope} = \frac{\Delta y}{h} = f(x_0, y_0). \qquad \text{So} \quad \Delta y = h f(x_0, y_0)$$

$$\frac{\Delta y}{h} = \operatorname{Tan}\theta$$

Euler method

x_0	y_0
$x_1 = x_0 + h$	$y_1 = y_0 + h f(x_0, y_0)$
$x_2 = x_1 + h$	$y_2 = y_1 + h f(x_1, y_1)$
$\vdots$	$\vdots$
$x_{n+1} = x_n + h$	$y_{n+1} = y_n + h f(x_n, y_n)$

EX

$$\frac{dy}{dx} = \underbrace{1 - x + 4y}_{f(x,y)} \qquad y(0) = 1$$

$x_0 = 0 \quad y_0 = 1$

Step size: $h = 0.1$

$x_1 = .1$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + (.1)\,[1 - 0 + 4 \cdot 1]$$

$$= 1.5$$

$$\boxed{(x_1, y_1) = (.1, 1.5)}$$

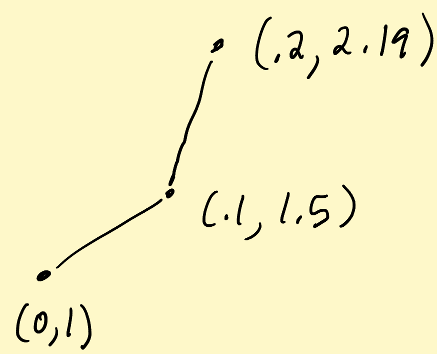
$x_2 = .2$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$= (1.5) + (.1)\left[1 - \underset{x_1}{(.1)} + 4\underset{y_1}{(1.5)}\right]$$

$$= 2.19$$

$$\boxed{(x_2, y_2) = (.2, 2.19)}$$



Use Excel!

n	x_n	y_n	$f(x_n, y_i)$	$\overbrace{y_n + h f(x_n, y_n)}^{y_{n+1}}$
0	0	1	$1-0+4\cdot 1=5$	1.5
1	.1	1.5		2.19
2	.2	2.19		

Fun fact

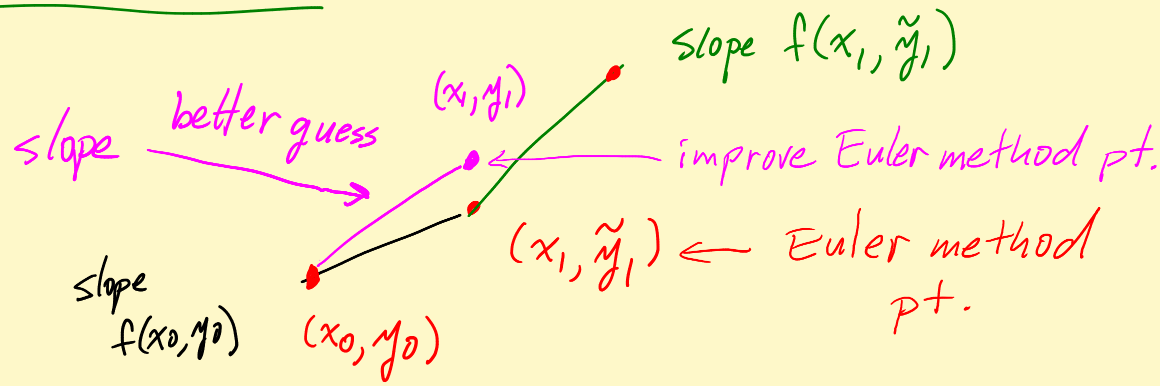
Easy problem: $\dfrac{dy}{dx} = f(x)$

Euler method: $y_N = y_0 + \sum_{k=0}^{N-1} f(x_n) \cdot h$ Riemann sum!

Rectangle method

$$\rightarrow y_0 + \int_{x_0}^{x_N} f(t)\,dt \quad \checkmark$$

Improved Euler method



Better guess = Ave : $\dfrac{f(x_0, y_0) + f(x_1, \tilde{y}_1)}{2}$.

Method $\qquad x_n = x_0 + nh \qquad \tilde{y}_{n+1} = y_n + h f(x_n, y_n)$

Step 1	Slope 1	Slope 2
	$f(x_n, y_n)$	$f(x_{n+1}, \tilde{y}_{n+1})$

Improved slope : $\dfrac{f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2}$

Next pt : $\left(x_{n+1},\ y_n + h \cdot \left[\dfrac{f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2} \right] \right)$

Summary

$$x_{n+1} = x_n + h$$

$$\tilde{y}_{n+1} = y_n + h f(x_n, y_n)$$ ← forget about this one in an

iteration

$$y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2}\right]$$

Fun facts $\quad \frac{dy}{dx} = f(x)$

Imp. Euler $\sim$ Trapezoid rule!

Runge-Kutta $\sim$ Simpson's rule