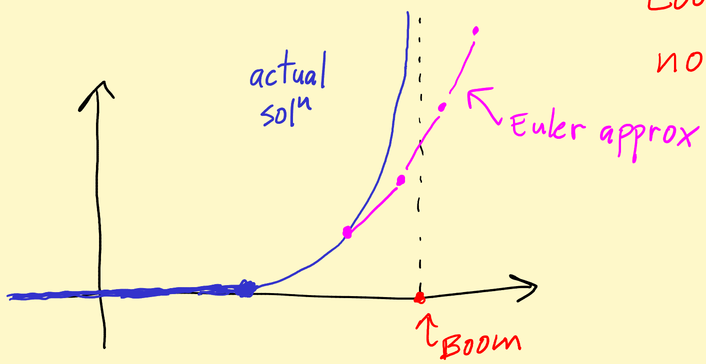
Lecture 13 Second order ODE (3.1)

MyLab HW 12
GS written HW 10, HW 12

Euler method fail: $\frac{dy}{dx} = y^2$ ← Recall sol^{n}s blow up in finite time. Looks like Euler method doesn't even notice!



F = ma is a 2nd order diff eqn!

Easiest 2nd order ODE: $\frac{d^2y}{dx^2} = -\sin x$

$$\frac{dy}{dx} = \int -\sin x \, dx = \cos x + C_1$$

$$y = \int \cos x + C_1 \, dx = \underbrace{\sin x}_{\text{particular sol}^n \text{ to } y'' = -\sin x} + \underbrace{c_1 x + C_2}_{\text{Gen}^l \text{ sol}^n \text{ to } y''=0}$$

(homogeneous eqn)

2nd order linear ODE:

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = D(x)$$

Standard form: Divide out by $A(x)$. Get

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \qquad (*)$$

where $P = \dfrac{B(x)}{A(x)}$, $Q = \dfrac{C(x)}{A(x)}$, $R = \dfrac{D(x)}{A(x)}$

↑ zeroes of $A(x)$ might cause trouble!

E. & U. Thm $(*)$ plus two intial conditions

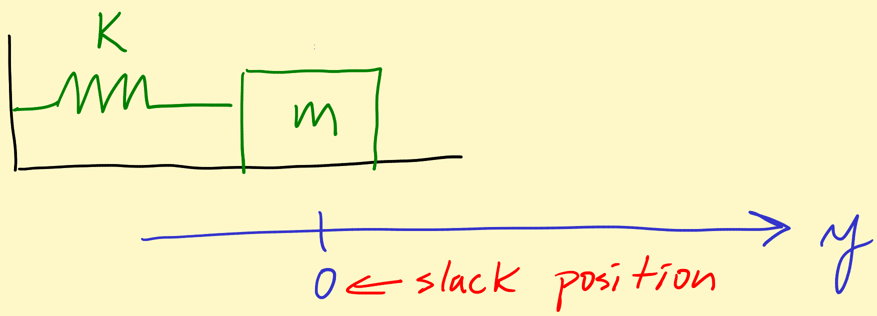
$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$

has a unique sol^n

(has ↑ sol^n exists (E.)) (unique ↑ (U.))

on any interval where $P(x), Q(x), R(x)$ are continuous.

EX: $y'' + 3y' + 2y = 0$ ← homog eqn



$F = ma$

$$\underbrace{-ky}_{\text{Hooke's law}} \; \underbrace{-cv}_{\text{friction}} = ma \qquad m\underbrace{\frac{d^2y}{dt^2}}_{a} + c\underbrace{\frac{dy}{dt}}_{v} + ky = 0$$

Could try "Reduction of order" $y'' + 3y' + 2y = 0$

x is missing! Let $v = \frac{dy}{dx}$.

Then $\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\underbrace{\frac{dy}{dx}}_{v} = v\frac{dv}{dy}$

ODE: $v\frac{dv}{dy} + 3v + 2y = 0$ ← "homogeneous" 1st order ODE! Ugh!

Euler's trick: Instead of this, guess that

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2e^{rx}$$

Plug into ODE and force it.

$$y'' + 3y' + 2y = 0$$

$$\left[r^2e^{rx}\right] + 3\left[re^{rx}\right] + 2\left[e^{rx}\right] = 0 \quad \text{want}$$

$$\underbrace{(r^2 + 3r + 2)}_{\text{this needs to be zero!}}\underbrace{e^{rx}}_{\text{never } 0} = 0 \quad \text{want}$$

$r^2 + 3r + 2 = (r+1)(r+2) = 0$ Roots: $r = -1, -2$

Got <u>two</u> sol^{n}s $y_1 = e^{-x}$, $y_2 = e^{-2x}$

Superposition principle for <u>linear</u> <u>homogeneous</u> eqns.

If y_1 and y_2 solve a 2nd order linear homog ODE, then so does $c_1 y_1 + c_2 y_2$.

<u>Why</u> Write $L[y] = y'' + Py' + Qy$

L is a <u>linear</u> operator, meaning that

$$L[\underbrace{c_1 y_1 + c_2 y_2}_{\text{so this is a sol}^n\text{ too.}}] = c_1 \underbrace{L[y_1]}_{\substack{=0 \\ \text{if } y_1 \text{ a sol}^n}} + c_2 \underbrace{L[y_2]}_{\substack{=0 \\ \text{if } y_2 \text{ a sol}^n}}$$

<u>Check</u>: $L[c_1 y_1 + c_2 y_2] = (c_1 y_1 + c_2 y_2)'' + P(c_1 y_1 + c_2 y_2)' + Q(c_1 y_1 + c_2 y_2)$

$$= \underline{c_1 y_1''} + \underline{c_2 y_2''} + \underline{c_1 P y_1'} + \underline{c_2 P y_2'} + \underline{c_1 Q y_1} + \underline{c_2 Q y_2}$$

$$= c_1 L[y_1] + c_2 L[y_2]$$

Aha! $c_1 e^{-x} + c_2 e^{-2x}$ is a sol^n.

Big question: Is it the gen$^{\ell}$ sol^{n}?

Equivalent question: Can I solve **any and all** initial value problems (IVP) $\begin{cases} y(x_0) = A \\ y'(x_0) = B \end{cases}$

for **any** choice of A's and B's?

(Equivalence follows from the E & U Thm)

Hmmm.

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

want

Use $x_0 = 0$. Want $\begin{cases} y(0) = c_1 e^0 + c_2 e^0 = A \\ y'(0) = -c_1 e^0 - 2c_2 e^0 = B \end{cases}$

$$\begin{cases} c_1 + c_2 = A \\ -c_1 - 2c_2 = B \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Cramer's rule

$$c_1 = \frac{\det\begin{bmatrix} A & 1 \\ B & -2 \end{bmatrix}}{\underbrace{\det\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{1\cdot(-2)-1\cdot(-1) = -1 \neq 0}}$$

Yay!

$$c_2 = \frac{\det\begin{bmatrix} 1 & A \\ -1 & B \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}$$

$$c_1 = \frac{-2A-B}{-1} = \underline{2A+B} \qquad c_2 = \frac{B+A}{-1} = \underline{-A-B}$$

<u>Key</u>: denom in Cramer's rule $\neq 0$. Can solve any and all IVP's!

Conclude $y = c_1 e^{-x} + c_2 e^{-2x}$ is the gen^l sol^n!

General problem

$$y = c_1 y_1 + c_2 y_2$$

$$y' = c_1 y_1' + c_2 y_2'$$

Use x_0.

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = A \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = B \end{cases}$$

want

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Want

$$\underbrace{\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\text{Wronskian} \quad W(x) = W[y_1, y_2]} \neq 0$$

EX

$$\det \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} = -2e^{-3x} + 1e^{-3x}$$

$$= -e^{-3x}$$ ← never zero!

Can solve any and all IVP's at any x_0!

Bummers $\quad ay'' + by' + cy = 0$

$$\underbrace{(ar^2 + br + c)}\, e^{rx} = 0$$

Bad cases: Only one r root.

Ouch! Complex roots.