Lecture 14 2^{nd} order linear ODE with constant coefficients (3.2)

MyLab HW 13

Existence and Uniqueness Thm (E & U) Suppose $P(x), Q(x), R(x)$ are continuous on an interval (a,b) and $x_0 \in (a,b)$.

Then the IVP

Inhomogeneous if $R \not\equiv 0$.

$$y'' + Py' + Qy = R \quad \text{with} \quad \begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$

$\underbrace{\text{has}}_{E}$ a $\underbrace{\text{unique}}_{U}$ sol^n that is twice continuously differentiable on (a,b).

Homogeneous problem: $y'' + P(x)y' + Q(x)y = 0$ (*)

Last time: If we find two solutions y_1 and y_2 to (*) with Wronskian

$$W(x_0) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \neq 0,$$

then $c_1 y_1 + c_2 y_2$ is the gen^l sol^n to (*).

Cool fact: Abel's formula shows that either

$W[y_1, y_2] \equiv 0$ or

never zero on (a,b).

Why: Suppose y_1 and y_2 are sol^{n}s to (*). Then

$$y_1 \cdot \left[y_2'' + P y_2' + Q y_2 = 0 \right]$$

$$- \quad y_2 \cdot \left[y_1'' + P y_1' + Q y_1 = 0 \right]$$

$$\underbrace{\left(y_1 y_2'' - y_2 y_1'' \right)}_{\frac{dW}{dx}} + P \underbrace{\left(y_1 y_2' - y_2 y_1' \right)}_{W} = 0$$

Aha!

$$\frac{dW}{dx} = \frac{d}{dx} \operatorname{Det} \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \frac{d}{dx} \left(y_1 y_2' - y_2 y_1' \right)$$

$$= y_1 y_2'' - y_2 y_1'' + \underbrace{y_1' y_2' - y_2' y_1'}_{0} \quad \checkmark$$

First order linear ODE: $\frac{dW}{dx} + PW = 0$

$$u = e^{\int P(x)\,dx} \qquad \underbrace{u \frac{dW}{dx} + uPW}_{[uW]'} = 0$$

So $uW = C$

$$W = C \cdot \frac{1}{u} = C \underbrace{e^{-\int P(x)\,dx}}_{\text{never zero}}$$

Abel's formula: $W = Ce^{-\int P(x)\,dx}$

Two case: 1) $C \neq 0$: W <u>never</u> zero

2) $C = 0$: $W \equiv 0$.

Important consequence: Only need to check W at <u>one</u> pt. Pick an easy one (like $x=0$)!

Meaning of $W(x_0) = 0$

Think about system

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = A & = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = B & = y_0' \end{cases}$$

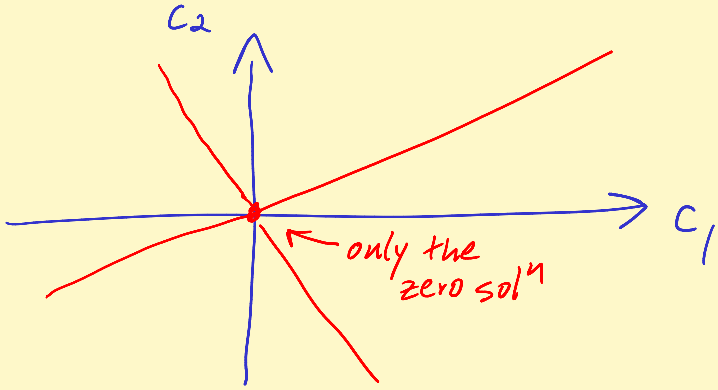
$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_2'(x_0) & y_2'(x_0) \end{bmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

If $\det \neq 0$, get unique sol^{n}s c_1, c_2 for any given choice of A, B

(via Cramer's rule).

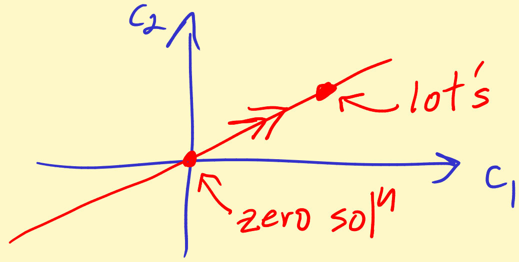
Determinant:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$



det $\neq 0$

When det $= 0$, two lines are the <u>same</u> line



MA 265 facts: $\mathbb{A}\vec{x} = 0$, $\mathbb{A}$ square matrix.

1) Solution space $\{\vec{x} : \mathbb{A}\vec{x} = \vec{0}\}$
is a <u>vector</u> space.

2) $\mathbb{A}\vec{x} = \vec{0}$ has the unique sol^n $\vec{x} = \vec{0}$
if and only if det $\mathbb{A} \neq 0$.

3) $\mathbb{A}\vec{x} = \vec{0}$ has non-zero vector sol^{n}s
if and only det $\mathbb{A} = 0$.

So, if $W(x_0) = 0$, then there are constants
a_1 and a_2 <u>not both zero</u> $\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq \vec{0} \right]$

such that
$$\begin{cases} a_1 y_1(x_0) + a_2 y_2(x_0) = 0 & (A) \\ a_1 y_1'(x_0) + a_2 y_2'(x_0) = 0 & (B) \end{cases}$$

Hmmm. Write $y = a_1 y_1 + a_2 y_2$

(A) means $\begin{cases} y(x_0) = 0 \\ y'(x_0) = 0 \end{cases}$ IVP
(B) means

Aha! The identically zero sol^n also satisfies this

$\exists$ of E & $\exists !$ $\Rightarrow$ $a_1 y_1 + a_2 y_2 \equiv 0$

Suppose $a_2 \neq 0$. Then $y_2 = \dfrac{-a_1}{a_2} y_1$

y_2 is a constant multiple of y_1

Fancy word: y_1 and y_2 are linearly dependent.

Consequence Only need to find two sol^{n}s y_1 & y_2 to $(*)$ that are "different" to get a gen^l sol^n $c_1 y_1 + c_2 y_2$.

EX: e^{-x} and e^{-2x} are "different"
e^{3x} and $2e^{3x}$ are not "different"

Last time $\quad ay'' + by' + cy = 0 \quad$ a, b, c const.

Tried $y = e^{rx}$:

$$\underbrace{(ar^2 + br + c)}_{\text{need } =0} \, e^{rx} = 0$$

never 0

Characteristic eqn $\quad \boxed{ar^2 + br + c = 0}$

Best case : 1) Two distinct real roots

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $b^2 - 4ac > 0$.

$c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is gen$^{\ell}$ sol^n

Bummer cases : $\quad$ 2) $\quad b^2 - 4ac = 0$

Only one root, $\quad r_1 = \frac{-b}{2a}$ ← real

Get one sol^n : $\quad y_1 = e^{r_1 x}$

Ouch! need a second linearly independent sol^n y_2

Euler : $\quad L[y] = ay'' + by' + cy$

$$L[e^{rx}] = \underbrace{(ar^2+br+c)}_{a(r-r_1)^2}e^{rx}$$

$\leftarrow r_1$ repeated root

Euler: differentiate this eqn $\frac{\partial}{\partial r}$

$$\underbrace{\frac{\partial}{\partial r}L[e^{rx}]}_{=L\left[\frac{\partial}{\partial r}e^{rx}\right]} = a\,2(r-r_1)e^{rx} + a(r-r_1)^2 xe^{rx}$$

Aha!

$$L[xe^{rx}]$$

Aha! Let $r=r_1$: $L[xe^{r_1x}] = 0$

The second sol^n: $y_2 = xe^{r_1x}$.

<u>Check</u> $W[y_1, y_2] \neq 0$ ✓ or

$$\frac{y_2}{y_1} = \frac{xe^{r_1x}}{e^{r_1x}} = x$$

$\leftarrow$ not constant! ✓

Bad case (2): Gen^l sol^n

$$y = c_1e^{r_1x} + c_2xe^{r_1x}$$

Case (3) : $b^2 - 4ac < 0$. Complex roots

$$r_1, r_2 = \alpha \pm \beta i$$

Euler: Who cares! Get complex sol^n

$$e^{(\alpha+\beta i)x} = e^{\alpha x + \beta x i} = e^{\alpha x} e^{\beta x i}$$

$$= e^{\alpha x}\left[\cos \beta x + i \sin \beta x \right]$$

Euler's identity

Got complex sol^n

$$\underline{e^{\alpha x} \cos \beta x} + i\, \underline{e^{\alpha x} \sin \beta x}$$

One complex sol^n yields two real sol^{n}s !

Real sol^n: $\quad y = c_1 \underbrace{e^{\alpha x} \cos \beta x}_{y_1} + c_2 \underbrace{e^{\alpha x} \sin \beta x}_{y_2}$

$\dfrac{y_2}{y_1} = \tan \beta x \leftarrow$ not constant. Got gen^l sol^n. ✓