Lecture 15 2^{nd} order linear ODE with const coeff, part 2.

MyLab HW 14

Euler: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ ← even when $z \in \mathbb{C}$!

$$e^{x+iy} = e^x \cdot e^{iy}$$
$$= e^x \left[1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \right]$$
$$= e^x \left[\underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right)}_{\cos y} + i \underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right)}_{\sin y} \right]$$

$$e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + i e^x \sin y$$

make this the definition

Last time $\quad ay'' + by' + cy = 0$ ← homog

Try $y = e^{rx}$. $\quad \underbrace{(ar^2 + br + c)}_{\text{need } =0} e^{rx} = 0$

Get $\quad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Case r_1, r_2 real and $\neq$ $\quad b^2 - 4ac > 0$

Gen^l Sol^n: $\quad y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case r_1, r_1 repeated root $\quad b^2 - 4ac = 0$

$$r_1 = \frac{-b}{2a}, \quad \text{Gen}^l \text{ sol}^n \quad y = c_1 \underbrace{e^{r_1 x}}_{y_1} + c_2 \underbrace{x e^{r_1 x}}_{y_2}$$

Case $r_1, r_2 = \alpha \pm \beta i$ complex roots $\quad b^2 - 4ac < 0$

Two real linearly independent sol^{n}s

$$\begin{cases} y_1 = e^{\alpha x} \cos \beta x \\ y_2 = e^{\alpha x} \sin \beta x \end{cases}$$

Gen^l sol^n $\quad y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$

Why complex case makes sense:

Complex sol^{n}s: $\quad Y(x) = u(x) + i v(x)$

Define $\quad Y'(x) = \lim_{h \to 0} \frac{Y(x+h) - Y(x)}{h} \leftarrow$ DQ

$$= \lim_{h \to 0} \left[(\text{DQ } u) + i (\text{DQ } v) \right]$$

$$= u'(x) + i v'(x)$$

Do it again: $\Upsilon''(x) = u''(x) + i v''(x)$

Key fact A complex sol^n $\Upsilon = u + iv$ yields two real sol^{n}s u and v.

Why: a, b, c are real and eqn is linear.

$$L[y] = ay'' + by' + cy$$

$$0 = L[\Upsilon] = a\Upsilon'' + b\Upsilon' + c\Upsilon$$

complex sol^n

$$= a[u'' + iv''] + b[u' + iv'] + c[u + iv]$$

$$= \underbrace{L[u]}_{=0} + i\,\underbrace{L[v]}_{=0}$$

So u, v are real sol^{n}s.

Key fact #2: $\frac{d}{dx}\left[e^{(\alpha+\beta i)x}\right] = (\alpha+\beta i)e^{(\alpha+\beta i)x}$

Write out both sides. Show equal ✓

and $\frac{d^2}{dx^2}\left[e^{(\alpha+\beta i)x}\right] = (\alpha+\beta i)^2 e^{(\alpha+\beta i)x}$

$$L[e^{rx}] = (ar^2 + br + c)\,e^{rx}$$ even when $r \in \mathbb{C}$.

When r is a complex root, get complex sol^n e^{rx}!

Get $$e^{(\alpha+\beta i)x} = e^{\alpha x} e^{\beta x i}$$
$$= e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$= \underbrace{e^{\alpha x}\cos\beta x}_{y_1} + i\,\underbrace{e^{\alpha x}\sin\beta x}_{y_2}$$

two real sol^{n}s

Wronskian? $W[y_1, y_2] =$

$$\det\begin{bmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x & \alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x \end{bmatrix} \leftarrow (\text{Row } 2) - \alpha\cdot(\text{Row } 1)$$

$$= \det\begin{bmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ -\beta e^{\alpha x}\sin\beta x & \beta e^{\alpha x}\cos\beta x \end{bmatrix}$$

$$= \beta e^{2\alpha x}\underbrace{(\cos^2\beta x + \sin^2\beta x)}_{=1} = \beta e^{2\alpha x}$$

never zero!

Great news! y_1 & y_2 form a gen^l sol^n.

Forget how we got here! Forget about the complex sol^n.

Hmmm. Picked <u>one</u> complex $\alpha+\beta i$ root.

Got two real sol^{n}s.

what if I had used the other one $\alpha-\beta i$ instead!

Get same answer; $e^{(\alpha-\beta i)x} = \underbrace{e^{\alpha x}\cos\beta x}_{y_1} - i\, e^{\alpha x}\sin\beta x$

$y_2 = -(\text{first } y_2)$

Leads to <u>same</u> gen^l sol^n $c_1 y_1 + c_2(-y_2) = c_1 y_1 + \tilde{c}_2 y_2$

Forget about second root two!

Do we really need Wronskian W? No!

$$\frac{y_2}{y_1} = \frac{e^{\alpha x}\sin\beta x}{e^{\alpha x}\cos\beta x} = \tan\beta x$$

not constant!

y_1 and y_2 are <u>linearly independent</u> and form a gen^l sol^n.

<u>Higher order eqns</u> $ay''' + by'' + cy' + dy = 0$

IVP

$$y(x_0) = y_0 \quad = A$$
$$y'(x_0) = y_0' \quad = B$$
$$y''(x_0) = y_0'' \quad = C$$

Play same game. Try $y = e^{rx}$

Get 3 roots of $ar^3 + br^2 + cr + d = 0$

r_1, r_2, r_3. Get $y = c_1 y_1 + c_2 y_2 + c_3 y_3$

IVP:

$$\begin{cases} y(x_0) = c_1 y_1(x_0) + \cdots + c_3 y_3(x_0) = A \\ y'(x_0) = c_1 y_1'(x_0) + \cdots + c_3 y_3'(x_0) = B \\ y''(x_0) = c_1 y_1''(x_0) + \cdots + c_3 y_3''(x_0) = C \end{cases}$$

want

Wronskian!

$$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) & y_3(x_0) \\ y_1'(x_0) & y_2'(x_0) & y_3'(x_0) \\ y_1''(x_0) & y_2''(x_0) & y_3''(x_0) \end{bmatrix} \neq 0$$

want

Facts 1) If $W(x_0) \neq 0$, we get gen^l sol^n

(via E & U thm).

2) Abel's formula holds: Consequently
W is either $\equiv 0$ or never zero.

3) (2) $\Rightarrow$ just need check $W(x_0) \neq 0$ at one x_0.
[pick a nice one like $x_0 = 0$ or 1.]

Need 3 "different" sol^{n}s to get a gen^l sol^n.

Need y_1, y_2, y_3 to be linear independent.

Need: If $c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0$,
then the c's all have to zero.