Lecture 16 Higher order linear ODE with constant coeff (3.3)

MyLab HW 15
GS HW 13, 14

$y_1, y_2, \dots, y_n$ are <u>linearly independent</u> on $[a,b]$ means:

$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \equiv 0$ on $[a,b]$

is <u>only</u> possible if all c's $= 0$.

<u>Note</u>: <u>Not</u> lin. ind. is called "linearly dependent."

Happens exactly when <u>one</u> of the fcns is a linear combination of the others.

<u>EX</u> Are x^2+x+1, x^2+x, x^2 lin. ind. on $(-\infty,\infty)$?

What if $c_1(x^2+x+1) + c_2(x^2+x) + c_3 x^2 \equiv 0$

$$\underbrace{[c_1 + c_2 + c_3]}_{\text{must } = 0} x^2 + \underbrace{[c_1 + c_2]}_{=0} x + \underset{\substack{\uparrow \\ c_1 = 0 \\ \uparrow \\ \#1}}{c_1} \equiv 0$$

#3: $c_3 = 0$ #2: $c_2 = 0$

All c's must $= 0$. ✓ lin. ind.

<u>EX</u> what about $\sin^2 x$, $\cos 2x$, 3?

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\sin^2 x + \frac{1}{2}\cos 2x - \frac{1}{2\cdot 3}\cdot 3 \equiv 0 \leftarrow \text{linearly dependent}$$

Const. Coeff linear ODE : n-th order

$$(a_n \neq 0) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y \equiv \underset{\text{homog}}{0}$$

Try $y = e^{rx}$

$$\underbrace{\left[a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right]}_{\text{need } = 0} \underbrace{e^{rx}}_{\text{never zero}} \equiv 0 \quad \text{want}$$

Need n lin. ind. sol^{n}s! Good news: n-th deg polys have n roots!

Best case Get n real distinct roots

$$r_1, r_2, \ldots, r_n$$

Gen^l sol^n $y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$

$$W = \det \begin{bmatrix} e^{r_1 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & \cdots & r_n e^{r_n x} \\ \vdots & & \\ r_1^{n-1} e^{r_1 x} & \cdots & r_n^{n-1} e^{r_n x} \end{bmatrix} \leftarrow n \times n$$

not zero. Got gen^l sol^n

EX Factor poly

$$r(r-2)(r-3)^3(r^2+2r+2)^2\underbrace{(r^2+1)}=0$$

$r=(r-0)$

$y_1=e^{0\cdot x}=\underline{1}$

$r=2$

$\underline{y_2=e^{2x}}$

$r=3,3,3$

$$3\left\{\begin{array}{l} e^{3x} \\ xe^{3x} \\ x^2e^{3x}\end{array}\right.$$

$r=-1\pm i$

repeated

$\underline{e^{-x}\cos x}$, $\underline{e^{-x}\sin x}$

$\underline{xe^{-x}\cos x}$, $\underline{xe^{-x}\sin x}$

$r=\pm i$

$=0\pm i$

$\alpha=0 \quad \beta=1$

$e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$

$\underline{\cos x}$, $\underline{\sin x}$

<u>Fact</u> Polys with <u>real</u> coeff can be factored into terms

1) $(r-r_0)^n$ (r_0 real)

<u>EX</u> $r=(r-0)^1$

$(r-7)$

$(r-1)^{13}$

2) and quadratic polys with complex roots (in pairs) $\alpha\pm\beta i$

$$(r^2+br+c)^n \qquad b^2-4c<0$$

<u>EX</u> (r^2+1) $\quad r=\pm i$

(r^2+2r+2)

$(r^2+2r+2)^{10}$

Simple real root case : $(r-r_0)^1$

Get sol^n $y = e^{r_0 x}$

Repeated real root $(r-r_0)^N$

Get
$$\left.\begin{array}{l} e^{r_0 x} \\ x e^{r_0 x} \\ \vdots \\ x^{N-1} e^{r_0 x} \end{array}\right\} N \text{ solutions!}$$

Euler trick

$$L[e^{rx}] = P(r)\, e^{rx} = (r-r_0)^N Q(r) e^{rx}$$

Diff w.r.t r :

$$\underbrace{\frac{\partial}{\partial r} L\left[e^{rx}\right]}\ = \text{vanishes to order } N-1 \text{ at } r_0$$

$$L\Big[\underbrace{\tfrac{\partial}{\partial r} e^{rx}}_{xe^{rx}}\Big] =$$

Set $r = r_0$.

Repeat!

Get $\frac{\partial^2}{\partial r^2} e^{rx} = x^2 e^{rx}$

Repeat $N-1$ times.

Complex simple roots

$(r^2 + br + c)$ $\quad b^2 - 4c < 0$

roots $\alpha \pm \beta i$

$$\text{Sol}^n \quad e^{(\alpha+\beta i)x} = \underbrace{e^{\alpha x} \cos \beta x} + i\, \underbrace{e^{\alpha x} \sin \beta x}$$

one complex sol^n yields <u>two</u> real sol^{n}s

Repeated complex roots $\quad (r^2+br+c)^N$

Complex sol^{n}s

$$\left.\begin{array}{l} e^{(\alpha+\beta i)x} \\ x\, e^{(\alpha+\beta i)x} \\ x^2\, e^{(\alpha+\beta i)x} \\ \vdots \\ x^{N-1}\, e^{(\alpha+\beta i)x} \end{array}\right\} \qquad \begin{array}{l} e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x \\ x e^{\alpha x}\cos\beta x, \ x e^{\alpha x}\sin\beta x \\ \vdots \\ x^{N-1}e^{\alpha x}\cos\beta x, \ x^{N-1}e^{\alpha x}\sin\beta x \end{array}$$

N complex sol^{n}s $\qquad\qquad$ $2N$ real sol^{n}s !

Factoring n-th degree polys with real coeff

<u>EX</u> $\quad r^4+8r^2+16=0 \qquad y^{(4)}+8y''+16y=0$

Aha! Quadratic in (r^2).

$$(\underbrace{r^2+4}_{r=\pm 2i \ \leftarrow \text{repeated}})^2 = 0$$

Roots $\quad 2i, 2i, -2i, -2i$

$\cos 2x, \ \sin 2x$

$x\cos 2x, \ x\sin 2x$

Gen'l sol'n $y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$

Need **4** initial conditions to pin down 4 c's

$$4 \checkmark \begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ y''(x_0) = y_0'' \\ y'''(x_0) = y_0''' \end{cases}$$

EX

$$r^6 + 5r^5 + 4r^4$$

$$r^4 \big(\underbrace{r^2 + 5r + 4}_{(r+4)(r+1)} \big)$$

$r = 0,0,0,0$

$$y = c_1 \cdot 1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-4x} + c_6 e^{-x}$$

Fact If a poly $P(r)$ vanishes at r_0 $[P(r_0) = 0]$, then $(r - r_0)$ divides the poly.

EX $r^3 - r^2 + r - 1 = P(r)$ See $P(1) = 0$.

So $r-1$ divides $P(r)$

$$
\begin{array}{r|l}
 & r^2 + 1 \\
\hline
r-1 & r^3 - r^2 + r - 1 \\
 & r^3 - r^2 \\
\hline
 & \qquad\quad r - 1 \\
 & \qquad\quad r - 1 \\
\hline
 & \qquad\quad 0
\end{array}
$$

So $P(r) = (r-1)\left[r^2 + 1\right]$

$$y = c_1 e^x + c_2 \cos x + c_3 \sin x$$

Operator notation: $D = \frac{d}{dx}$

$$ay'' + by' + cy = aD^2y + bDy + cy$$

$$= \left[aD^2 + bD + c\right]y$$