# Lecture 18 The method of undetermined coefficients (3.5)

MyLab HW 17

<u>Big fact</u>: $(\dagger)$ $A(x)y'' + B(x)y' + C(x)y = F(x)$ ← non-homogeneous

$(*)$ $A(x)y'' + B(x)y' + C(x)y = 0$ ← homogeneous

<u>Gen$^l$ Sol$^n$</u> to $(\dagger)$ is

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{Gen}^l \text{ Sol}^n \text{ to homog eqn } (*)} + y_p$$

$y_p$ ← One <u>particular</u> sol$^n$ to $(\dagger)$

Call this $y_c$

"complementary sol$^n$"

<u>Why</u>: $L[y] = Ay'' + By' + Cy$ is a <u>linear</u> operator.

Suppose $y_p$ is a particular sol$^n$ to $(\dagger)$.

Suppose $Y$ is another. Then

$$
\begin{array}{rl}
- & L[y_p] = F \\
 & L[Y] = F \\
\hline
\end{array}
$$

$$L[Y] - L[y_p] = F - F$$

$$L[\underbrace{Y - y_p}] = 0$$

$Y - y_p$ solves homog $(*)$. So

$\Psi - y_p = c_1 y_1 + c_2 y_2$. Aha! $\Psi = c_1 y_1 + y_2 + y_p$

Last thing to check: All such things solve (†).

Easy $L[c_1 y_1 + c_2 y_2 + y_p] = c_1 L[y_1] + c_2 L[y_2] + L[y_p]$

$= 0 + 0 + F$ ✓

EX: $y'' + 4y = e^{-2x}$

Step 1: Solve homog eqn $y'' + 4y = 0$

$r^2 + 4 = 0 \quad r = \pm 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

Step 2 Find one particular sol$^n$ however you can!

Aha! Guess $\boxed{y_p = Ae^{-2x}}$

Step 3 Make it work!

want

Want $y_p'' + 4y_p = e^{-2x}$

$(-2)^2 A e^{-2x} + 4(Ae^{-2x}) = e^{-2x}$

$(4A + 4A) e^{-2x} = e^{-2x}$

Need $8A = 1$. $A = 1/8$.

Got $y_p = \frac{1}{8} e^{-2x}$

Step 4: Gen$^l$ sol$^n$ to $(+)$:

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} e^{-2x}$$

## Method of undetermined coeff

$$ay'' + by' + cy = F(x)$$ ← something special

| $F(x)$ | Try $y_p =$ ← trial sol$^n$ |
|---|---|
| $e^{rx}$ | $Ae^{rx}$ |
| $x^n$ or any poly of deg $n$ | $A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$ (gen$^l$ poly of degree $n$) |
| $\cos wx$<br>$\sin wx$<br>$a \cos wx + b \sin wx$ | $A \cos wx + B \sin wx$ |
| $e^{rx} \cos wx$<br>$e^{rx} \sin wx$ | $Ae^{rx} \cos wx + Be^{rx} \sin wx$ |

| | |
|---|---|
| $x^n e^{rx} \cos wx$ | $(A_n x^n + \cdots + A_1 x + A_0) e^{rx} \cos wx +$ |
| $x^n e^{rx} \sin wx$ | $(B_n x^n + \cdots + B_1 x + B_0) e^{rx} \sin wx$ |

Rule: But, if any single piece of the "trial sol$^n$" from the table solves the homog prob, try $y_p = x^s$ (trial sol$^n$ from table) where $s$ is the smallest positive integer so that no term in the new trial sol$^n$ solves the homog eqn.

EX $\quad y'' + y = \cos x \longleftarrow$ resonance

homog sol$^n$: $\quad r^2 + 1 = 0 \qquad r = \pm i$

$$y_c = c_1 \cos x + c_2 \sin x$$

Table: Trial sol$^n$ is $\quad y_p = A \cos x + B \sin x$.

Ouch! Plug in. Can only get $= 0$, not $\cos x$.

New trial sol$^n$: $\quad y_p = x^s (A \cos x + B \sin x)$

pieces $= Ax^s \cos x, \quad Bx^s \sin x$

Smallest $s$ so these do solve homog eqn. $\boxed{s=1}$.

Correct try:

$$y_p = x\left(A\cos x + B\sin x\right)$$

Do it:

$$y_p' = (A\cos x + B\sin x) + x(-A\sin x + B\cos x)$$

$$y_p'' = 0\cdot(A\cos x + B\sin x) + 2\cdot 1\cdot(-A\sin x + B\cos x) + x(-A\cos x - B\sin x)$$

$$(uv)'' = u''v + 2u'v' + uv''$$

Plug into (+). Collect terms.

$$\underbrace{\left[-2A\sin x + 2B\cos x - x(A\cos x + B\sin x)\right]}_{y_p''} + \underbrace{\left[x(A\cos x + B\sin x)\right]}_{y_p}$$

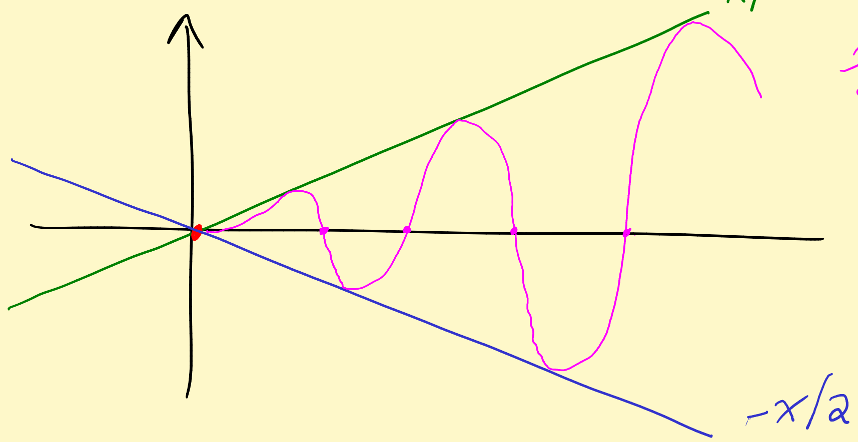
Miracle: $\quad -2A\sin x + 2B\cos x \overset{\text{want}}{=} \cos x$

Need $-2A = 0, \quad 2B = 1$

$A = 0, \quad B = \frac{1}{2}$

Get $y_p = x\left(0 \cdot \cos x + \frac{1}{2} \cdot \sin x\right) = \frac{x}{2} \sin x$

Gen$^l$ sol$^n$: $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x$




Ouch!

<u>EX</u> $y'' + 4y' + 4y = e^{-2x}$

Homog $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

Repeated $r = -2, -2$

$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

Trial sol$^n$ $y_p = Ae^{-2x}$ ← ouch! Solves homog

Next, try $y_p = x(Ae^{-2x}) = Axe^{-2x}$ ← also solves homog

Next: $y_p = x^2(Ae^{-2x}) = Ax^2e^{-2x}$ Safe!

( $s=2$ is smallest $s$. )

Finally, plug in $y_p = Ax^2e^{-2x}$ and make work.

Fun thing Same example:

$$(D^2+4D+4)y = e^{-2x}$$

$$(D+2)^2 y = e^{-2x}$$

What differential operator annihilates RHS $e^{-2x}$?

Answer: $(D+2)$ does.

$$(D+2)\left[(D+2)^2 y\right] = \underbrace{(D+2)e^{-2x}}_{0}$$

$$(D+2)^3 y = 0$$

$y$ solves this eqn

$r=-2,-2,-2$

$$y = \underbrace{c_1e^{-2x} + c_2xe^{-2x}}_{\text{homog sol}^n \text{ to original}} + \underbrace{c_3x^2e^{-2x}}_{\text{this must be a}}$$

Aha!

good choice for a particular $sol^n$!

HWK prob $\quad y^{(5)} + 5y^{(4)} - y = 17$

poly deg zero

Ouch! 5-th order. Can't find roots of homog characteristic eqn, but

Method of undet coeff says try

$$y_p = A \leftarrow \text{gen}^l \text{ poly deg zero.}$$

Safe! Check that it does not solve homog eqn.

Pieces of $\quad (A_1 x + A_0)e^{2x}\cos 3x + (B_1 x + B_0)e^{2x}\sin 3x$

$$\begin{cases} A_1 x e^{2x}\cos 3x \\ A_0 e^{2x}\cos 3x \\ B_1 x e^{2x}\sin 3x \\ B_0 e^{2x}\sin 3x \end{cases}$$