Lecture 19 Variation of parameters (3.5)

MyLab HW 18

Rule: But, if any single piece of the "trial sol^n" from the table solves the homog prob, try

$$y_p = x^s \left(\text{trial } sol^n \text{ from table}\right)$$

where s is the smallest positive integer so that no term in the new trial sol^n solves the homog eqn.

EX $\quad y^{(4)} + 2y'' + y = x\cos x$

$$(D^2+1)^2\, y = x\cos x \leftarrow (\text{poly of deg } 1)\cdot(\text{Cos or Sin})$$

Table: Trial sol^n

$$y_p = (A_1x + A_0)\cos x + (B_1x + B_0)\sin x$$

pieces: $A_0\cos x$, $B_1 x\sin x$ ouch! solve homog

$$(r^2+1)^2 = 0$$

$r = \pm i$ repeated

$$e^{ix}, \quad xe^{ix}$$

$\cos x \quad \sin x \quad x\cos x \quad x\sin x$

Homog sol^n

$$c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x$$

Correct guess: $x^s\left[(A_1x + A_0)\cos x + (B_1x + B_0)\sin x\right]$

$s=1$ doesn't work: Piece $A_0 x\cos x \leftarrow$ solves homog

$s=2$ does work!

Right thing to plug in and force

$$y_p = x^2\left[(A_1x+A_0)\cos x + (B_1x+B_0)\sin x\right]$$

Variation of parameters

$$\underbrace{y'' + P(x)y' + Q(x)y}_{L[y] \text{ not constant coeff,}} = \underset{\uparrow \text{ F not special}}{F(x)}$$

L[y] not constant coeff, or F not special

Step 1 Solve homog eqn $L[y] = 0$

Get gen^{l} sol^{n} $y_c = c_1 y_1 + c_2 y_2$

Step 2 Get a particular sol^{n}

$$y_p = u_1 y_1 + u_2 y_2$$

letting parameters vary! (c_1, c_2 in y_c)

where u_1 and u_2 are functions satisfying

$$\boxed{u_1' = \frac{-y_2 F}{W[y_1, y_2]} \qquad u_2' = \frac{y_1 F}{W[y_1, y_2]}}$$

Danger! Using formula requires F to be from Standard form eqn.

Why Try $y_p = u_1 y_1 + u_2 y_2$

Plug in ODE, force

(Hmmm. Need to determine two fcns u_1, u_2.

Plugging in ODE only gives one eqn to determine two fcns.)

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{=0} + u_1 y_1' + u_2 y_2'$$

Aha! $=0$ (sneaky) can be my 2nd eqn!

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug in ODE

$$(E)\quad \underbrace{\left[u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''\right]}_{y_p''} + P\underbrace{\left[u_1 y_1' + u_2 y_2'\right]}_{y_p'} + Q\underbrace{\left[u_1 y_1 + u_2 y_2\right]}_{y_p} \overset{\text{want}}{=} F$$

color: $u_1 L[y_1] = 0$

color: $u_2 L[y_2] = 0$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & \text{(sneaky)} \\ u_1' y_1' + u_2' y_2' = F & \text{(E)} \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Cramer's rule!

$$u_1' = \frac{\det\begin{bmatrix} 0 & y_2 \\ F & y_2' \end{bmatrix}}{\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{-y_2 F}{W} \checkmark$$

$$u_2' = \frac{\det\begin{bmatrix} y_1 & 0 \\ y_1' & F \end{bmatrix}}{W} = \frac{y_1 F}{W} \checkmark$$

<u>EX</u> $\underbrace{1 \cdot y'' + y}_{\text{const coeff}} = \underbrace{\sec x}_{\text{not special!}} \longleftarrow F(x) = \sec x$

$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$

Get $y_p = u_1 y_1 + u_2 y_2$ where ...

first calculate $W[y_1, y_2] = \det\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$= \cos^2 x + \sin^2 x = \underline{1}$

$$u_1' = \frac{-y_2 F}{W} = \frac{-(\sin x)(\sec x)}{1} \leftarrow \sec x = \frac{1}{\cos x}$$

$$= -\frac{\sin x}{\cos x} = -\tan x$$

Integrate! $u_1 = \int -\tan x \, dx = -\ln|\sec x| \underbrace{+C}_{\text{skip } +C}$

Just want a u_1 that works.
Skip $+c$.

$$(\underbrace{\sim + c}_{u_1}) y_1 = \sim + \underbrace{c_1 y_1}_{\text{solves homog}}$$

$$u_1 = -\ln\underbrace{|\sec x|}_{|\cos x|^{-1}} = \ln|\cos x|$$

$$u_2' = \frac{y_1 F}{W} = \frac{(\cos x)(\sec x)}{1} = 1$$

Integrate: $u_2 = \int 1 \, dx = x$ (no $+c$)

Done: $y_p = \underbrace{(\ln|\cos x|)}_{u_1}\underbrace{(\cos x)}_{y_1} + \underbrace{(x)}_{u_2}\underbrace{(\sin x)}_{y_2}$

Gen^l sol^n:

$$y = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} + \underbrace{(\cos x)\ln|\cos x| + x \sin x}_{y_p}$$

EX Euler eqn!

$$\underbrace{x^2 y'' - 4xy' + 4y}_{\text{not const. coeff}} = -2x^2$$

special ✓

Euler: Homog solⁿ. Try $y = x^r$

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

Plug in $x^2\left[r(r-1)x^{r-2}\right] - 4x\left[rx^{r-1}\right] + 4\left[x^r\right] = 0$ want

$$\Big(\underbrace{r(r-1) - 4r + 4}_{\text{need } = 0}\Big)x^r = 0$$

$$r^2 - r - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$r = 1, 4$$

Get $y_1 = x^1 = x, \quad y_2 = x^4$

Linearly independent? $\dfrac{y_2}{y_1} = \dfrac{x^4}{x} = x^3$ not constant! Indep. ✓

$$y_c = c_1 x + c_2 x^4$$

$$W = \det\begin{bmatrix} x & x^4 \\ 1 & 4x^3 \end{bmatrix} = 4x^4 - x^4 = 3x^4$$

What? W vanishes at $x=0$! Is Abel wrong?

Aha! <u>Standard form.</u> Divide by x^2

$$y'' - \frac{4x}{x^2} y' + \frac{4}{x^2} y = \frac{(-2x^2)}{x^2}$$

$$y'' - \underbrace{\frac{4}{x}}_{\text{bad}} y' + \underbrace{\frac{4}{x^2}}_{\text{bad}} y = \underline{\underline{-2}}$$

F(x)!

bad at $x=0$ where $W=0$.

Safe on intervals not containing $x=0$.

$$u_1' = \frac{-(x^4)(-2)}{3x^4} = \frac{2}{3} \qquad u_1 = \frac{2}{3}x$$

$$u_2' = \frac{(x)(-2)}{3x^4} = -\frac{2}{3}x^{-3} \qquad u_2 = -\frac{2}{3}\left(\frac{1}{-3+1}x^{-3+1}\right) = \frac{1}{3}x^{-2}$$

Get $y_p = \left(\frac{2}{3}x\right)(x) + \left(\frac{1}{3}x^{-2}\right)(x^4) = x^2$

Gen'l Sol'n $y = c_1 x + c_2 x^4 + x^2$

Important fact $\quad L[y] = F_1 + F_2 + F_3 \qquad (+)$

Get $\quad L[y_{p_j}] = F_j$

Part. sol^n for $(+)$ $= y_{p_1} + y_{p_2} + y_{p_3}$

because L is linear.