Lecture 20 Forced vibrations (3.6)

MyLab HW 19

GS HW 16, 17, 18

Undamped free spring-mass: $m\ddot{y} + ky = 0 \qquad \omega = \sqrt{\frac{k}{m}}$

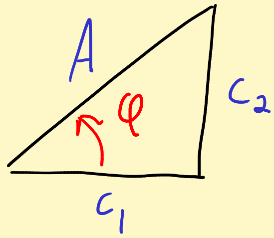
Solution $y = c_1 \cos \omega t + c_2 \sin \omega t = A \cos(\omega t - \varphi)$

want

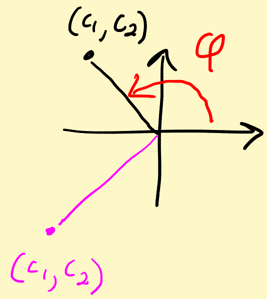
Initial conditions: $y(0) = c_1 = y_0$

$y'(0) = \omega c_2 = y_0'$

Easy!



$$\begin{cases} \varphi = \tan^{-1}\left(\frac{c_2}{c_1}\right) \\ A = \sqrt{c_1^2 + c_2^2} \end{cases}$$



or (1): $y(0) = A \cos(-\varphi) = \boxed{A \cos \varphi = y_0}$

$y'(t) = -A\omega \sin(\omega t - \varphi)$

(2): $y'(0) = -A\omega \sin(-\varphi) = \boxed{A\omega \sin \varphi = y_0'}$

$$\frac{(2)}{(1)} = \frac{A\omega \sin \varphi}{A \cos \varphi} = \boxed{\omega \tan \varphi = \frac{y_0'}{y_0}}$$

Get $\varphi = \tan^{-1}\left(\frac{y_0'}{\omega y_0}\right)$

Now (1): $A = \frac{y_0}{\cos \varphi}$

Ave on Exam 1 : 82

Forced vibrations

Undamped case $\quad m\ddot{y} + ky = \underbrace{F(t)}_{F_0 \cos \omega t}$ ← "input"

<u>Step 1</u> Solve homog :

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t = A \cos(\omega_0 t - \varphi)$$

where $\quad \omega_0 = \sqrt{\frac{k}{m}}$. $\qquad \left(r = \pm \sqrt{\frac{k}{m}}\, i \right)$

<u>Step 2</u> Get part. sol^n $\; y_p$.

<u>Case 1</u> $\quad \omega \neq \omega_0$

$$y_p = A \cos \omega t + B \sin \omega t$$

Safe! These don't solve homog.

$$y_p' = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$y_p'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$m\left[-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \right] + k\left[A \cos \omega t + B \sin \omega t \right]$$

$$\left[-mA\omega^2 + kA \right] \cos \omega t + \left[-mB\omega^2 + kB \right] \sin \omega t \overset{\text{want}}{=} F_0 \cos(\omega t)$$

$$\underbrace{\qquad\qquad}_{F_0} \qquad \underbrace{\qquad}_{0} \leftarrow \text{So } B=0$$

$$A = \frac{F_0}{K - m\omega^2} = \frac{F_0}{m\left(\frac{K}{m} - \omega^2\right)} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$\uparrow\ \omega_0^2$

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \longleftarrow \text{"output"}$$

↑ Whoa! When ω close to ω_0, this gets big!

Gen^l sol^n

$$y = A\cos(\omega_0 t - \varphi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

EX

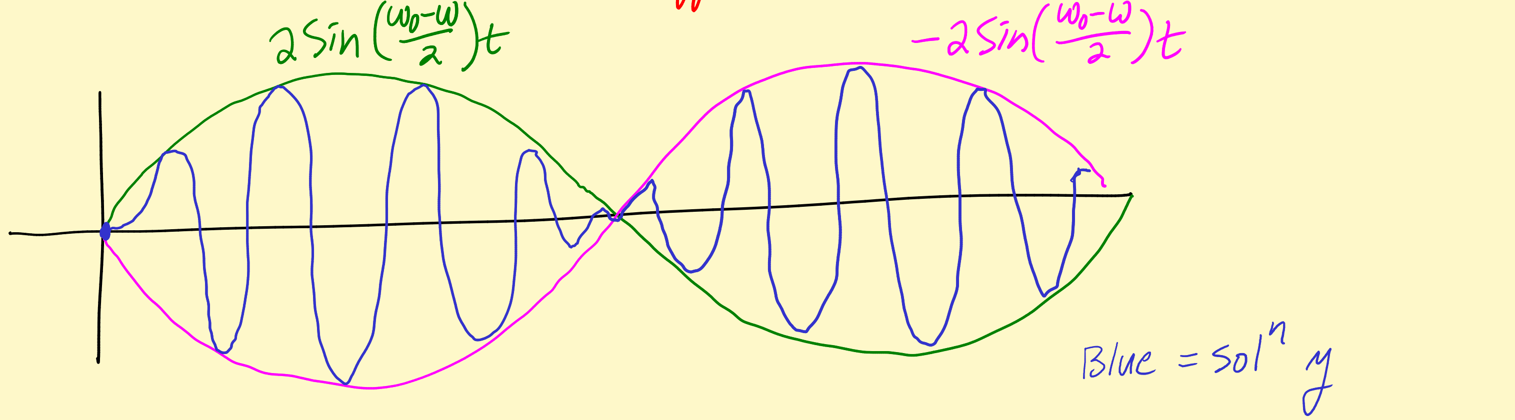
$$y = \cos \omega t - \cos \omega_0 t$$

$$= 2 \sin\left(\frac{\omega_0 - \omega}{2}\right)t \ \sin\left(\frac{\omega_0 + \omega}{2}\right)t \quad \leftarrow \text{trig identity}$$

Case ω close to ω_0

↑ small — "slow wiggler"

↑ close to ω_0 ← fast wiggler

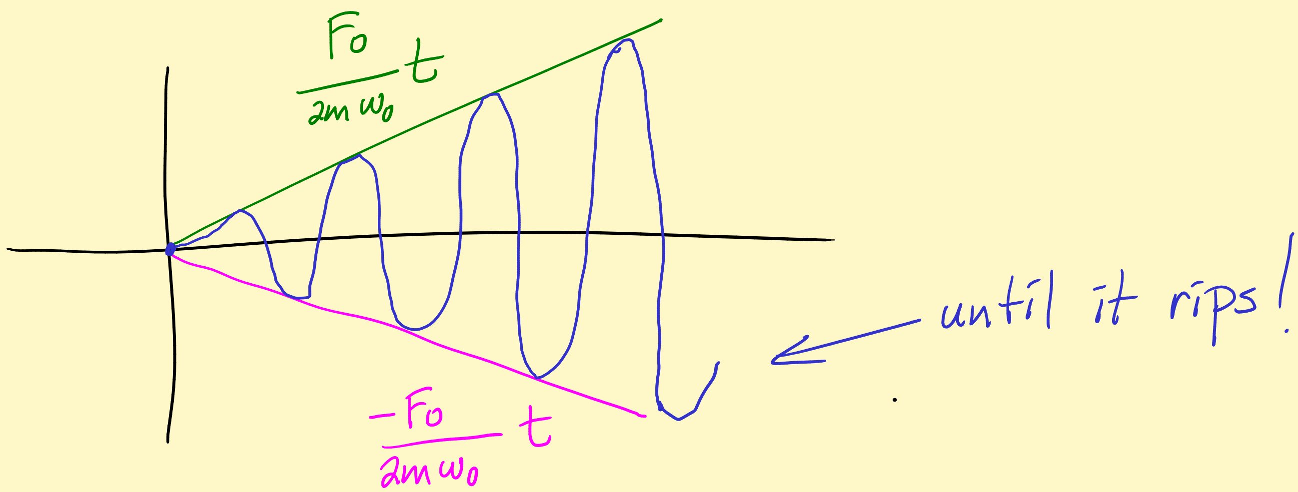


Case $\omega = \omega_0$ Resonance!

Try: $y_p = t\left[A\cos\omega_0 t + B\sin\omega_0 t\right]$

$$m y'' + k y = F_0 \cos \omega_0 t \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases} \qquad \text{Get} \quad \boxed{y = \frac{F_0}{2m\omega_0} t \sin \omega_0 t}$$



Damping $\qquad m y'' + c y' + k y = F_0 \cos \omega t$

$(c > 0)$

Step 1 Solve homog. $\qquad m r^2 + c r + k = 0$

$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

Underdamped $\qquad c^2 - 4km < 0$

$$r = \underbrace{\frac{-c}{2m}}_{\text{negative}} \pm \underbrace{\frac{\sqrt{|c^2 - 4km|}}{2m}}_{\mu} i$$

homog sol^n $\quad y_c = c_1 e^{-\frac{c}{2m}t} \cos \mu t + c_2 e^{-\frac{c}{2m}t} \sin \mu t$

fizzles out!

"transient sol^n"

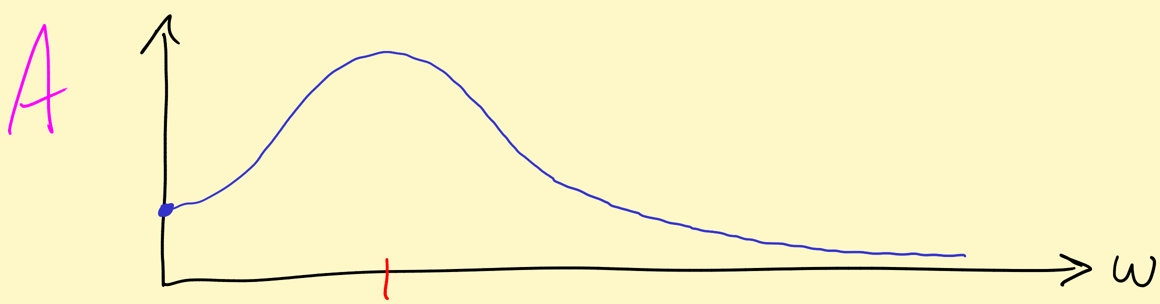
$$= a\, e^{-\frac{c}{2m}t} \cos(\mu t - \varphi)$$

Step 2 $\quad y_p = A \cos \omega t + B \sin \omega t$

Safe. Doesn't solve homog even if $\omega = \omega_0 = \sqrt{\frac{k}{m}}$.

Get

$$y_p = \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}}_{\text{"amplitude of the response"} = A} \cos(\omega t - \eta)$$



as close to a "resonant frequency" as you can get

Not equal to ω_0

Other two cases $\quad$ Overdamped: $c^2 - 4km > 0$

Get two roots r_1, r_2 both negative.

Transient sol^n $y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1, r_2 < 0$

$\longrightarrow 0$ as $t \to \infty$

Critically damped: $c^2 - 4km = 0$

$$r = -\frac{c}{2m} = -\frac{c}{2m}, \; -\frac{c}{2m} \text{ (repeated root)}$$

$$y_c = c_1 e^{-\frac{c}{2m}t} + c_2 \underbrace{t e^{-\frac{c}{2m}t}}$$

also $\to 0$ as $t \to \infty$.

Transient too.