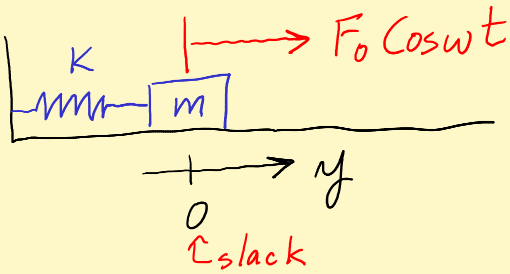
Lecture 21 Forced Vibrations (3.6 part 2)

MyLab HW 20



$$m\ddot{y} + c\dot{y} + ky = F_0 \cos \omega t$$

Over $c^2 - 4km > 0$

Critical $c^2 - 4km = 0$

Under $c^2 - 4km < 0$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

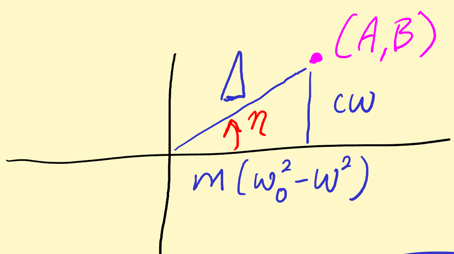
$$mr^2 + cr + k = 0$$

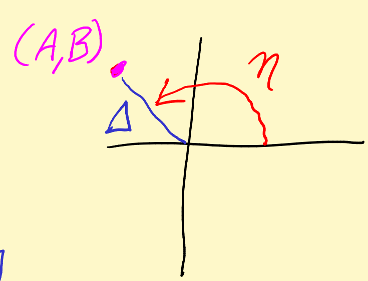
roots $-\frac{c}{2m} \pm \frac{\sqrt{c^2-4km}}{2m}$

$$y = \underbrace{\begin{cases} c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} & (O) \\ c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t} & (C) \\ a e^{-\frac{c}{2m}t} \cos(\mu t - \delta) & (U) \end{cases}}_{\text{transient sol}^n\text{s } y_c \to 0 \text{ as } t \to \infty} + \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \eta)}_{\text{particular sol}^n \; y_p}$$

(U) : $\mu = \dfrac{\sqrt{|c^2 - 4km|}}{2m}$ $\qquad r = -\frac{c}{2m} \pm \mu i$

$$y_p = \underbrace{m(\omega_0^2 - \omega^2)}_{A} \cos \omega t + \underbrace{c\omega}_{B > 0} \sin \omega t$$





$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}$$

Amplitude of response $= \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}}$

$\longrightarrow 0$ as $\omega \to \infty$

Phase shift of response: $\eta = \operatorname{Tan}^{-1}\dfrac{c\omega}{m(\omega_0^2-\omega^2)} \longrightarrow 0$

as $\omega \to \infty$

Question Which input frequency yields largest response amplitude?

Sol^n $\dfrac{F_0}{\sqrt{}}$ is max when $\sqrt{}$ is min, and

$\sqrt{}$ is min when $\underbrace{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}_{\text{4-th deg poly in } \omega \,=\, P(\omega)}$ is min.

Fresh calc. prob!

$$P'(\omega) = 2m^2(\omega_0^2-\omega^2)(-2\omega) + 2c^2\omega$$

Crt. pts. $-2\omega\left[2m^2(\omega_0^2-\omega^2) - c^2\right] = 0$

$\omega = 0$, $\quad (\omega_0^2-\omega^2) = \dfrac{c^2}{2m^2}$

$$\omega^2 = \underbrace{\omega_0^2 - \frac{c^2}{2m^2}}_{} =$$

Cases, <0, $=0$, >0

$= \omega_0^2 - \underbrace{\frac{k}{m}}_{=\omega_0^2}\cdot\frac{m}{k}\cdot\frac{c^2}{2m^2}$

$= \omega_0^2\left(1 - \frac{c^2}{2km}\right)$

Properties of $P(\omega)$

1) $P(\omega) > 0$ for all ω.

2) Coeff of ω^4 is pos.

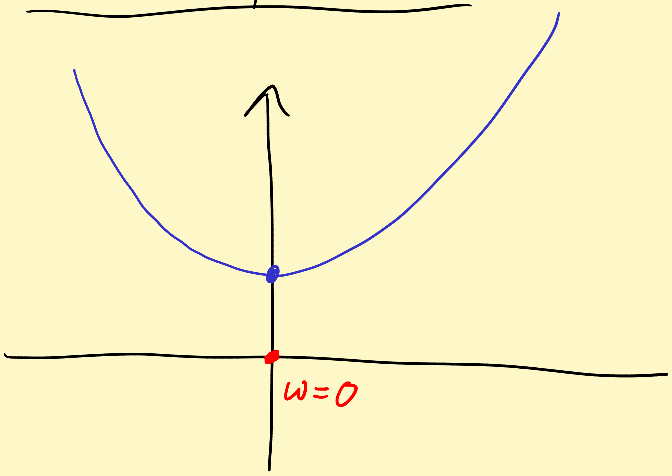
So $P(\omega) \nearrow \infty$ as $\omega \to \pm\infty$.

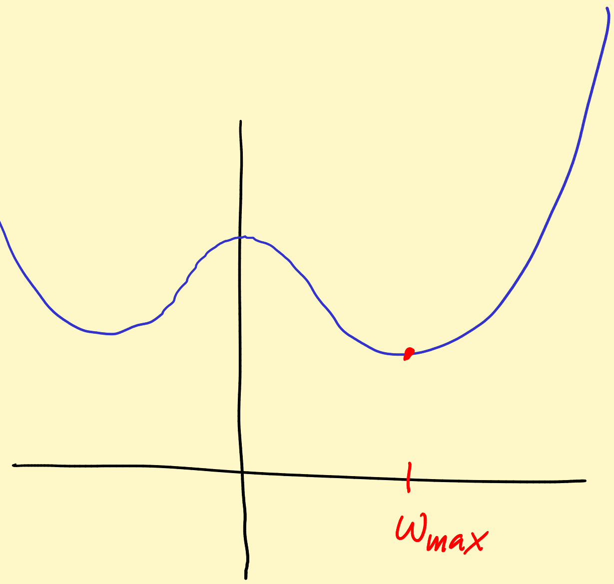
3) $P'(\omega)$ is 3^{rd} degree, so at most 3 crt pts.

Hmmm. Maybe 3 distinct, or
1 simple, 1 double root or
1 triple root

3) $P(\omega)$ is symmetric about vertical axis

Possible pictures



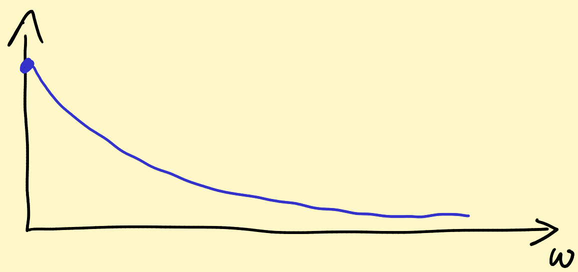


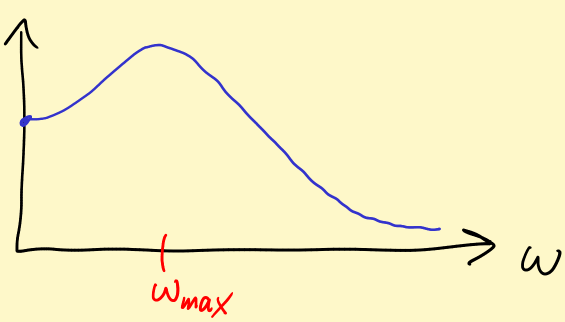
Symm $\Rightarrow$ Only these are possible

Fresh. calc. Get $\omega_{max} = \omega_0 \sqrt{\underbrace{1 - \frac{c^2}{2mk}}_{\text{when this is } > 0}}$

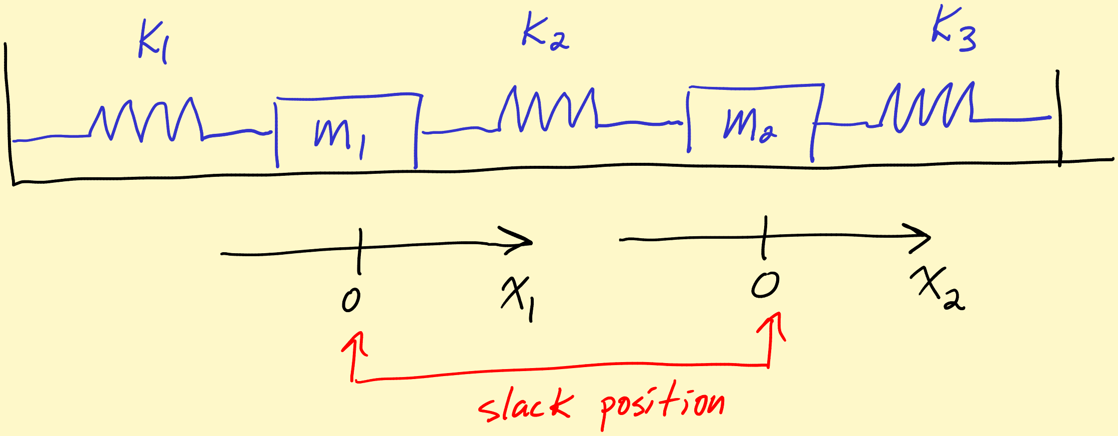
"Practical resonance"

Possible pictures of response curve





Fun problem (Chapter 4)



think $x_2 > x_1$ stretched

For m_1: $\quad m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$

For m_2: $\quad m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$

Same $+, -$'s in other case $x_2 < x_1$.

System of ODE, of 2 second order ODEs.

Assume $k_1 = k_2 = k_3 = 1 = m_1 = m_2$

(1) $\quad \ddot{x}_1 = -2x_1 + x_2 \quad \Leftarrow \quad \boxed{x_2 = \ddot{x}_1 + 2x_1}$

(2) $\quad \ddot{x}_2 = x_1 - 2x_2$

Plug box into (2): $\quad \dfrac{d^2}{dt^2}\Big[\underbrace{\ddot{x}_1 + 2x_1}_{x_2}\Big] = x_1 - 2\Big[\underbrace{\ddot{x}_1 + 2x_1}_{x_2}\Big]$

Aha! Get 4-th order linear ODE with const coeff!

$$x_1^{(4)} + 2\ddot{x}_1 = x_1 - 2\ddot{x}_1 - 4x_1$$

$$x_1^{(4)} + 4\ddot{x}_1 + 3x_1 = 0$$

(Method of elimination)

$$r^4 + 4r^2 + 3 = 0$$

$$(r^2+1)(r^2+3) = 0$$

Roots $r = \pm i, \pm\sqrt{3}\, i$

$$x_1 = c_1 \cos t + c_2 \sin t + c_3 \cos\sqrt{3}\, t + c_4 \sin\sqrt{3}\, t$$

Hmmm. $x_2 = \ddot{x}_1 + 2x_1$ (box)

$$x_2 = c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}\, t - c_4 \sin\sqrt{3}\, t$$

To pin down $\underline{\underline{4}}$ c's, need $\underline{\underline{4}}$ initial cond.

$$\begin{cases} x_1(0) = A_1 \\ x_2(0) = A_2 \end{cases} \qquad \begin{cases} \dot{x}_1(0) = B_1 \\ \dot{x}_2(0) = B_2 \end{cases}$$

Do these determine c's? Yes! Via a "wronskian" style argument.

Basic solutions: $c_1 = 1$, rest $= 0$

$c_2 = 1$, rest $= 0$

etc.

$c_1 = 1.$
rest $= 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \cos t \end{pmatrix} \leftarrow \text{slow, parallel}$$

$c_3 = 1$
rest $= 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \sqrt{3}\, t \\ -\cos \sqrt{3}\, t \end{pmatrix} \leftarrow \text{fast, opposing}$$