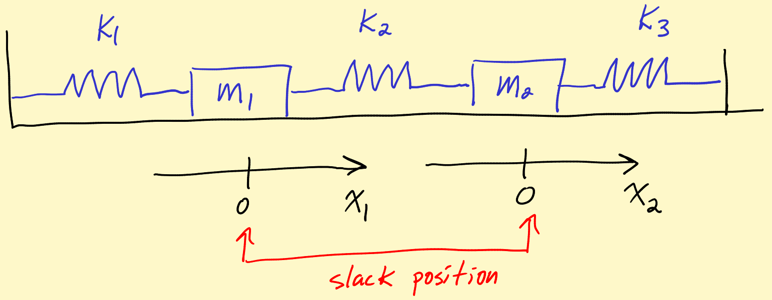
Lecture 22 Systems of ODE (4.1)

MyLab HW 21



$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases}$$

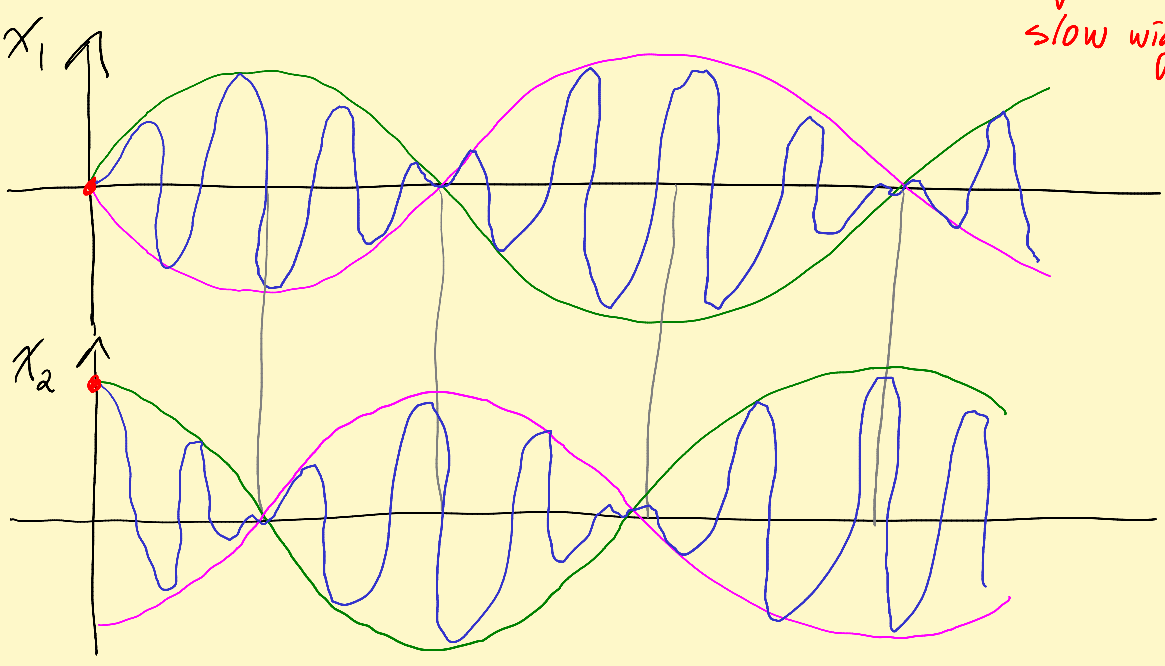
$$\begin{cases} x_1 = c_1 \cos t + c_2 \sin t + c_3 \cos\sqrt{3}\,t + c_4 \sin\sqrt{3}\,t \\ x_2 = c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}\,t - c_4 \sin\sqrt{3}\,t \end{cases}$$

Modes of oscillation: $c_k = 1$, rest $= 0$. $k = 1, 2, 3, 4$

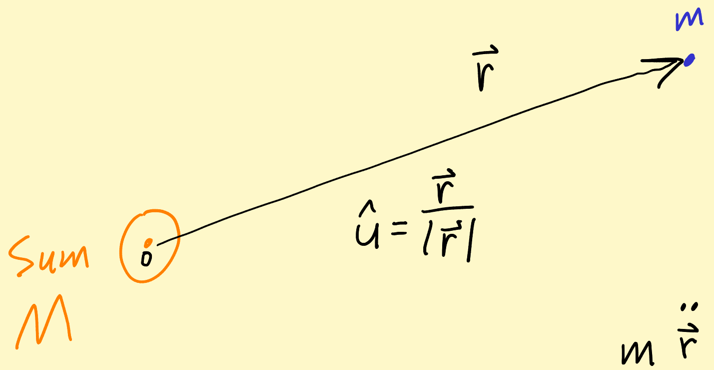
EX $\quad x_1(0) = 0 \qquad x_2(0) = 1$

$\qquad \dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$

Get
$$\begin{cases} x_1 = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{3}\,t = \sin\left(\frac{\sqrt{3}-1}{2}t\right)\sin\left(\frac{\sqrt{3}+1}{2}t\right) \\ x_2 = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{3}\,t = \underbrace{\cos\left(\frac{\sqrt{3}-1}{2}t\right)}_{\text{slow wiggler}}\ \underbrace{\cos\left(\frac{\sqrt{3}+1}{2}t\right)}_{\text{fast}} \end{cases}$$



Prob

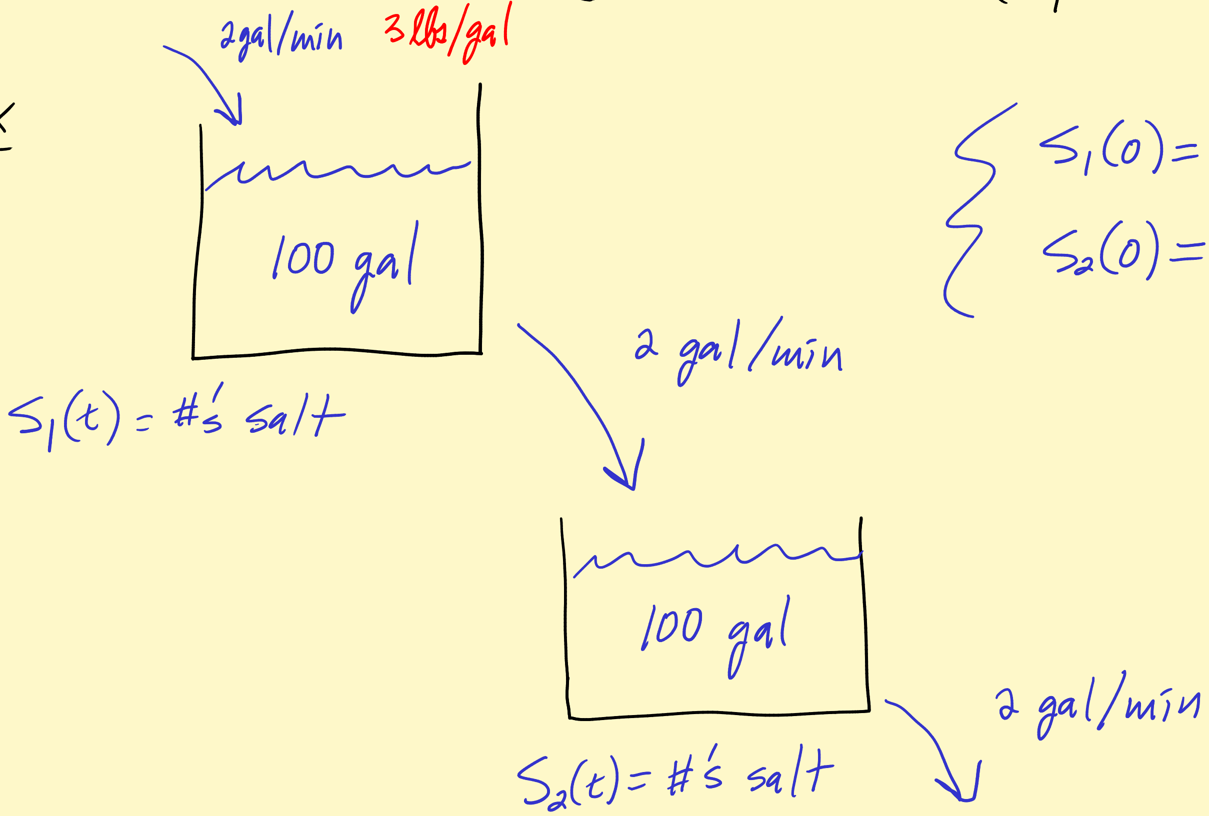


$$\vec{r} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$$

$$m \ddot{\vec{r}} = \frac{GMm}{|\vec{r}|^2} \left(\frac{-\vec{r}}{|\vec{r}|} \right)$$

$$\begin{cases} m \ddot{x}_1 = \dfrac{GMm}{(x_1^2 + x_2^2 + x_3^2)} \left(\dfrac{-x_1}{\sqrt{(\ \ )}} \right) \\ m \ddot{x}_2 = \cdots \\ m \ddot{x}_3 = \dfrac{-GMm\, x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \end{cases}$$

EX



$$\begin{cases} S_1(0) = 0 \\ S_2(0) = 0 \end{cases}$$

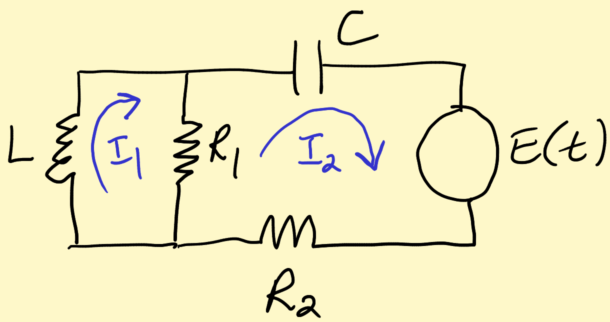
$$\frac{dS_1}{dt} = (\text{Rate in}) - (\text{Rate out}) = (2\,\text{gal/min})(3\,\text{lbs/gal}) - (2\,\text{gal/min})\left(\frac{S_1(t)}{100\,\text{gal}} \right)$$

$$\frac{dS_2}{dt} = (2\,gal/min)\left(\frac{S_1(t)}{100\,gal}\right) - (2\,gal/min)\left(\frac{S_2(t)}{100}\right)$$

$$\begin{cases} \dfrac{dS_1}{dt} = 6 - \dfrac{1}{50} S_1 \\[2ex] \dfrac{dS_2}{dt} = \dfrac{1}{50} S_1 - \dfrac{1}{50} S_2 \end{cases}$$

$\leftarrow$ linear, simple ODE in S_1
Solve it!
Plug $S_1(t)$ into eqn 2.
Solve it.

EX



Voltage drops

RI

$L\dfrac{dI}{dt}$

$\dfrac{1}{C}Q$

(A) $\quad L\dfrac{dI_1}{dt} + R_1(I_1 - I_2) = 0 \quad \leftarrow$ Kirchoff's

(B) $\quad \dfrac{1}{C}Q - E(t) + R_2 I_2 + R_1(I_2 - I_1) = 0$

Aha! Take $\frac{d}{dt}$(B): Use $\dfrac{dQ}{dt} = I_2$.

(B′) $\quad \dfrac{1}{C} I_2 + R_2\dfrac{dI_2}{dt} - R_1\left(\dfrac{dI_2}{dt} - \dfrac{dI_1}{dt}\right) = E'(t)$

(A) & (B′) system we can solve!

First order systems say it all!

EX $$y'' + P(x)y' + Q(x)y = F(x) \qquad (*)$$

Standard method to convert $(*)$ to a first order system.

$$\left.\begin{aligned} u_1 &= y \\ u_2 &= y' \end{aligned}\right\} \quad \text{Big idea!}$$

[$u_3 = y''$ if this were 3^{rd} order]

Step 1 Always $\longrightarrow$ $u_1' = (y)' = y' = u_2$

Step 2 (Use ODE) $u_2' = (y')' = y''$ ← Use ODE here

$$= -P\underset{u_2}{y'} - Q\underset{u_1}{y} + F$$

$$= -Pu_2 - Qu_1 + F$$

$$\begin{cases} u_1' = u_2 \\ u_2' = -Qu_1 - Pu_2 + F \end{cases} \qquad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\vec{u}\,' = \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}$$

$$\vec{u}' = A(x)\vec{u} + \vec{F}$$ ← looks so easy this way!

EX

$$\begin{cases} x_1' = 4x_1 + 2x_2 & (A) \\ x_2' = 3x_1 - x_2 & (B) \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\vec{x}$$

Method of elimination: Use (A): $\boxed{x_2 = \frac{1}{2}x_1' - 2x_1}$ box #1

Plug box #1 into (B)

$$\Big(\underbrace{\tfrac{1}{2}x_1' - 2x_1}_{x_2}\Big)' = 3x_1 - \Big(\underbrace{\tfrac{1}{2}x_1' - 2x_1}_{x_2}\Big)$$

$$\boxed{x_1'' - 3x_1' - 10x_1 = 0}$$

$$r^2 - 3r - 10 = 0$$

$$(r+2)(r-5) = 0$$

$$\boxed{\underline{x_1 = c_1 e^{-2t} + c_2 e^{5t}}}$$

Use box #1 to get $x_2 = \frac{1}{2}x_1' - 2x_1$

$$= \tfrac{1}{2}\left(-2c_1 e^{-2t} + 5c_2 e^{5t}\right) - 2\left(c_1 e^{-2t} + c_2 e^{5t}\right)$$

$$x_2 = -3c_1 e^{-2t} + \tfrac{1}{2} c_2 e^{5t}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{5t}}$$

Hmmm. Sol^{n}s look like $c_1 \vec{x}_1 + c_2 \vec{x}_2$

where $\vec{x}_j = \vec{b}_j e^{r_j t}$ $\quad j = 1, 2$

How to turn $\begin{cases} \ddot{x}_1 = 2x_1 - x_2 & (A) \\ \ddot{x}_2 = x_1 - 2x_2 & (B) \end{cases}$ into a 4x4 system

$u_1 = x_1$

$u_2 = \dot{x}_1$

$u_3 = x_2$

$u_4 = \dot{x}_2$

$$\begin{cases} u_1' = \dot{x}_1 = u_2 \\ u_2' = \ddot{x}_1 = 2u_1 - u_3 & \leftarrow \text{Use (A)} \\ u_3' = \dot{x}_2 = u_4 \\ u_4' = \ddot{x}_2 = u_1 - 2u_3 & \leftarrow \text{Use (B)} \end{cases}$$

$$\vec{u}\,' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \vec{u}$$

4x4 first order linear system with const. coeff