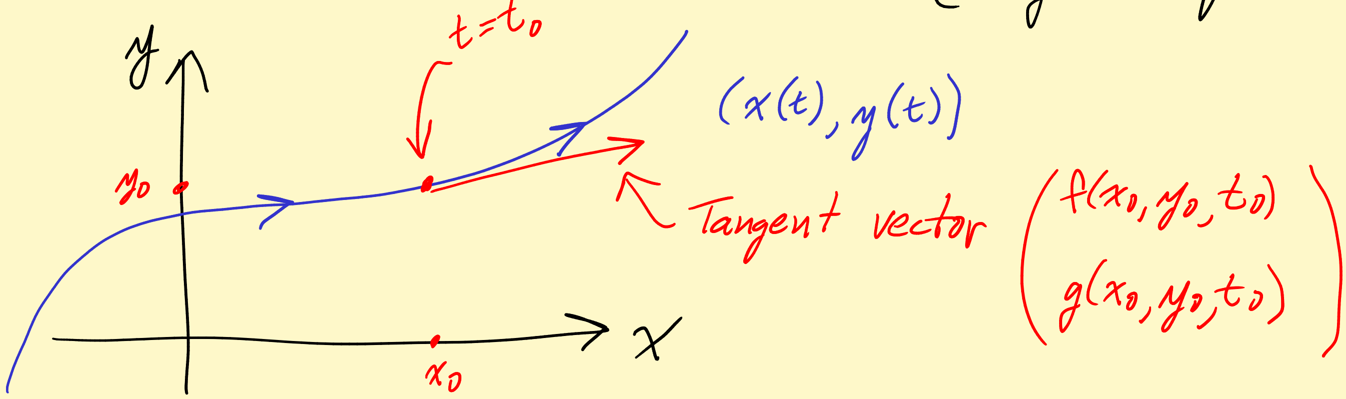
Lecture 23 Systems (4.2)

MyLab 22
GS HW 19, 20

$$(*)\begin{cases} \dfrac{dx}{dt} = f(x,y,t) \\[2ex] \dfrac{dy}{dt} = g(x,y,t) \end{cases}$$

Want fcns $x(t), y(t)$.

I.V.P. $\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$



E & U Thm: If f, g and their first partial derivatives are continuous on 3-D rectangle (x,y,t) containing (x_0, y_0, t_0), then solutions to $(*)$ exist on interval $(t_0-\delta, t_0+\delta)$ and the sol^n to the IVP is unique. ($\delta > 0$ might be small)

Linear systems

$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)\,x + p_{12}(t)\,y & + q_1(t) \\[2ex] \dfrac{dy}{dt} = p_{21}(t)\,x + p_{22}(t)\,y & + q_2(t) \end{cases}$$

homogeneous

non-homog.

IVP $\begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$

Improved E & U thm: Sol^n exists and is unique on largest interval where all p's are continuous.

EX Linear first order 2X2 homogeneous system with constant coeff.

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 & \text{(A)} \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 & \text{(B)} \end{cases}$$

Note $x_1 \equiv 0$, $x_2 \equiv 0$ clearly is a sol^n.

It is the unique sol^n with $\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases}$.

Method of elimination: Solve (A) for x_2.

Plug what you get into (B). Get 2nd order linear ODE in x_1 with const. coeff. Solve it.

Get $x_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

Plug into $x_2 = \cdots$ eqn, get

$$x_2 = c_1 A_1 e^{r_1 t} + c_2 A_2 e^{r_2 t}.$$

Notation:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 A_1 \end{pmatrix} e^{r_1 t} + \begin{pmatrix} c_2 \\ c_2 A_2 \end{pmatrix} e^{r_2 t}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ A_1 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} 1 \\ A_2 \end{pmatrix} e^{r_2 t}$$

Ques What are r's and vectors $\begin{pmatrix} 1 \\ A_1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ A_2 \end{pmatrix}$?

Ans Eigenvalues and eigenvectors of the obvious 2x2 matrix $\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$.

$$\begin{cases} x_1' = 5x_1 - x_2 \\ x_2' = 3x_1 + x_2 \end{cases}$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'}_{\vec{x}} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{\vec{x}' = \mathbb{A} \vec{x}}$$

Big idea Try sol^{n}s of form $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{rt} = \vec{a} e^{rt}$

$$\begin{cases} x_1 = a_1 e^{rt} \\ x_2 = a_2 e^{rt} \end{cases} \qquad\qquad \vec{x} = \vec{a} e^{rt}$$

Want

$$\begin{cases} \dfrac{d}{dt}\left(a_1 e^{rt}\right) = 5a_1 e^{rt} - a_2 e^{rt} \\[2ex] \dfrac{d}{dt}\left(a_2 e^{rt}\right) = 3a_1 e^{rt} + a_2 e^{rt} \end{cases} \qquad\qquad \left(\vec{a} e^{rt}\right)' = \mathbb{A} \vec{a} e^{rt}$$

$$\begin{cases} r a_1 e^{rt} = 5a_1 e^{rt} - a_2 e^{rt} \\ r a_2 e^{rt} = 3a_1 e^{rt} + a_2 e^{rt} \end{cases} \qquad\qquad r \vec{a} e^{rt} = \mathbb{A} \vec{a} e^{rt}$$

Cancel e^{rt}'s!

$$\begin{cases} r a_1 = 5a_1 - a_2 \\ r a_2 = 3a_1 + a_2 \end{cases} \qquad\qquad \underset{\text{eigenvalue}}{r} \vec{a} = \mathbb{A} \vec{a}$$

$\vec{a}$ eigenvector

Don't want all the a's to be zero. (We know the zero sol^n.)

$$\begin{cases} 0 = (5-r)a_1 - a_2 \\ 0 = 3a_1 + (1-r)a_2 \end{cases} \qquad\qquad \vec{0} = \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} \vec{a}$$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$$

Hmmm. Cramer's rule:

$$a_1 = \frac{\det\begin{bmatrix} 0 & -1 \\ 0 & 1-r \end{bmatrix}}{\det\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix}} = 0$$

$$a_2 = \frac{\det\begin{bmatrix} & 0 \\ & 0 \end{bmatrix}}{\det\begin{bmatrix} & \\ & \end{bmatrix}} = 0$$

To get a non-zero $\vec{a}$, Cramer's has to bomb!

Must have $\boxed{\det\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = 0}$

$$\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0}$$

Need $(5-r)(1-r) + 3 = 0$

$$r^2 - 6r + 8 = 0$$

$$(r-4)(r-2) = 0$$

$r = 2, 4$ eigenvalues.

Next, find eigenvectors for each r.

For $r=2$ $\quad (\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$

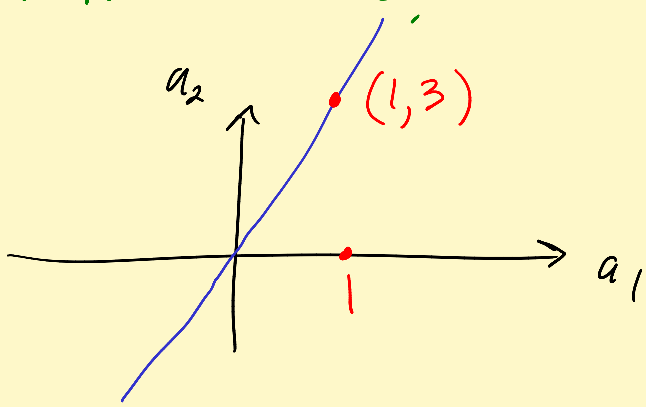
$r=2$

$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \vec{0}$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \vec{0}$$

One eqn in two unknowns! $\quad 3a_1 - a_2 = 0$

$a_2 = 3a_1$



Lots of non-zero vector sol^{n}s! Pick a nice one.

Nice one: $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Found sol^n $\quad \vec{x} = \vec{a}\, e^{rt} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

$$\begin{cases} x_1 = e^{2t} \\ x_2 = 3e^{2t} \end{cases} \quad \checkmark$$

<u>For $r=4$</u>: Get $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

<u>Superposition principle</u> If $\vec{x}_1$ and $\vec{x}_2$ solve a linear homog system, so do linear combos.

So $\quad c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$

looks like the gen^{l} sol^{n}. Is it?

E & U Thm: Can I solve any and all IVPs?

$$\begin{cases} x_1(0) = A \\ x_2(0) = B \end{cases}$$

$$c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2\cdot 0} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4\cdot 0} \overset{\text{want}}{=} \begin{pmatrix}A\\B\end{pmatrix}$$

$$\underbrace{\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}}_{\text{Cramer's rule}}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$$

$$\det = 1\cdot 1 - 3\cdot 1 = -2 \neq 0$$

Yes! Can solve all IVPs!